# Math 252:01 — Spring 2002
# MTh3 SEC-212
# Prof. Bumby

## Review Problems

In order to make it easier to find relevant sections of the text while practicing, these problems will be arranged by chapter. There is no guarantee that problems on the final exam will be grouped in any way.

**Chapter 1. Solving a single first order equation.** The course has emphasized the ability to **recognize** solutions of differential equations by various methods. Although the exam will concentrate on finding exact solutions, you should have developed sufficient experience with theoretical, graphical, and numerical methods to become suspicious when your calculations yield unlikely results. An outline of a correct method of solution is not enough; you need sufficient command of the method to produce correct answers. This review will be limited to problems suitable for an exam, but preparation for the exam should use the computational problems to include a review the techniques that check the accuracy of the solution.

**Separable equations**. The main class of equations that we are able to solve exactly are the **separable** equations, in which $f(y, t)$ is a product of a function of $y$ and a function of $t$. These are solved by **separating** the equation to get an expression depending only on $y$ multiplied by $dy$ equal to an expression depending only on $t$ times $dt$. The integrals of these expressions must differ by an **arbitrary constant**. This introduces a parameter into the solution. It is often possible to solve the resulting equation for $y$ as an explicit function of $t$ contain a parameter **somewhere** (usually no longer simply as a "constant of integration"). You should remember this **process** without trying to express it as a formula. Equations that seem similar often have solutions that are very different.

In section 1.2, exercises 5–24 (aiming for a general solution) and 25–34 (aiming for the unique solution of an initial value problem) give some practice. On quiz #1, you were asked for the **general solution** of

$$\frac{dy}{dt} = \frac{3(y-1)}{t-3}.$$

On quiz #2, you were asked for the **general solution** of

$$\frac{dy}{dt} = \frac{2ty}{t^2+1}$$

Elsewhere on that quiz, you used the slope field to sketch the solution with $y(0) = 1$. You should be able to use the formula to get an exact solution that could be compared to the the sketch. On the first exam, problem #1 asked for the general solution of

$$\frac{dy}{dt} = (y-1)^2(7-6t).$$

and the solution with $y(0) = 5/4$.

Here are more equations

$$\begin{aligned}\frac{dy}{dt} &= 7y - 3y^2\\ \frac{dy}{dt} &= te^t\\ \frac{dy}{dt} &= \frac{5y}{t} \quad (t > 0)\\ \frac{dy}{dt} &= \frac{2y}{t^2} \quad (t > 0)\\ \frac{dy}{dt} &= (t+1)(y^2+1)\end{aligned}$$

Two of these formulas are not defined when $t = 0$. When this line is removed from the $(t, y)$ plane, the plane falls into two **halfplanes**. To emphasize that the solutions in those two sets are independent, we have selected one of them. A careful solution will make use of this restriction.

For the equations that allow $t = 0$, the three solutions **(a)** with $y(0) = -1$, **(b)** with $y(0) = 0$, and **(c)** with $y(0) = 1$, should be found. In addition, for all equations, you should find the three solutions **(d)** with $y(1) = -1$, **(e)** with $y(1) = 0$, and **(f)** with $y(1) = 1$

**Linear equations**. An equation of the form

$$\frac{dy}{dt} = g(t)y$$

is separable. The solution is

$$\begin{aligned}\int \frac{dy}{y} &= \int g(t)\,dt\\ \ln y &= G(t) + C\\ y &= e^{G(t)}e^C\\ &= Ke^{G(t)}\end{aligned}$$

where $G(t)$ is any function with $G'(t) = g(t)$ and $K = e^C$. This assumes $y > 0$ and leads to $K > 0$, but the final formula is easily seen to be a solution for any constant $K$.

There are other ways to discover the form of the solution. The textbook ignores the relation to separable equations, while introducing the idea of an **integrating factor**. The quantity

$$\mu(t) = e^{-G(t)}$$

has the property that

$$\frac{d}{dt}\bigl(\mu(t)y(t)\bigr) = \mu(t)y'(t) - g(t)\mu(t)y(t) = \mu(t)\bigl(y'(t) - g(t)y(t)\bigr).$$

This is zero if $y(t)$ satisfies

$$\frac{d}{dt}y(t) = g(t)y(t),$$

so the solution of the equation is $\mu(t)y(t) = K$. This is equivalent to the previous result. The key property of linear equations is the way that solutions are parameterized. Note that the operation $L$ sending $y(t)$ to

$y'(t) - g(t)y(t)$ is **linear**. That is, multiplying $y(t)$ by a constant or taking a sum of functions $y(t)$ has the same effect on the result. From this, it follows that constant multiples of solutions of the equation we are studying are also solutions. If you find one nonzero solution, its constant multiples are also solutions, and there are enough of them to be able to satisfy any initial conditions.

This approach also applies to **inhomogeneous** linear equations.

$$y'(t) - g(t)y(t) = h(t).$$

Multiplying by the integrating factor converts this into

$$\frac{d}{dt}\big(\mu(t)y(t)\big) = \mu(t)h(t).$$

Integrating both sides solves the equation. Again, the process is easier to remember and to use than any formula you might derive to express it. When you solve for $y$, the result has the form of **a particular solution an arbitrary constant multiple of the solution of the corresponding homogeneous equation**. This is a consequence of the linearity of the operator $L$. In many cases, it is easier to guess a particular solution than it is to follow the process using the integrating factor. One form of systematic guessing is called **the method of undetermined coefficients**. It consists of selecting a finite list of functions that are likely to be needed to express a particular solution, and setting up the equations that say that the application of $L$ to a linear combination of these leads to the given $h(t)$.

Exercise 1-14 of Section 1.8 have solutions with simple expressions. You should experiment with different methods of solving the equation in order to find which approach you find easiest. Here are a few more equations.

$$\frac{dy}{dt} = 5y - 15$$

$$\frac{dy}{dt} = y + t - 2$$

$$\frac{dy}{dt} = y\tan t + 2\sin t \quad (-\pi/2 < t < \pi/2)$$

$$\frac{dy}{dt} = -\frac{2yt}{1+t^2} + 4t$$

$$\frac{dy}{dt} = \frac{3y}{t} + 4 - 3t$$

**Phase lines and bifurcation diagrams** are an important first step for work done later in the course. The equations studied here are **autonomous**, i.e, of the form $dy/dt = f(y)$. In particular, they are separable. This leads to solutions that equate a function of $y$ with $(t + C)$. This says that translating a solution in the $t$ direction always gives another solution. To get solutions giving $y$ as a function of $t$, one must solve this equation. However, this level of detail obscures what may be the most important feature of the equation. The validity of this method of solution is confined to domains on which $f(y) \neq 0$, but if $f(c) = 0$, the constant function $y(t) = c$ is a solution of the equation. All the solutions **between** two such **stationary values** look the same. The most important properties of a solution taking values in this interval is whether they are increasing or decreasing. This can be found by evaluating $f(y)$ at a representative point in the interval (it **cannot change sign** on the interval): the differential equation itself asserts that $f(y) > 0$ corresponds to increasing solutions and $f(y) < 0$ corresponds to decreasing solutions.

For autonomous equations, only a **phase line** coordinatized by $y$ needs to be shown, and the nature of the solutions is described on this line by an arrow showing the sign of $g(y)$. This shows the **direction of**

**motion** of solutions. The equation must be solved to see how time is measured by the motion on a solution, but the direction of motion is contained in the equation itself.

A graph of $g(y)$ includes information about the sign of the function, and various techniques used to draw the curve have direct interpretations on the phase line. In particular, if $g'(y) \neq 0$ at a point where $g(y) = 0$, $g$ is increasing, so it must be negative for smaller $y$ and positive for larger $y$. This says that the phase line arrows near this point are directed away from the point. Such a stationary point is called a **source** with solutions of the equation appearing to originate there when $t \to -\infty$. Similarly, $g'(y) < 0$ describes a **sink** with solutions approaching this point as $t \to +\infty$.

Having mastered individual phase lines, you are ready to look at infinitely many phase lines arranged in a **bifurcation diagram**, which is drawn in a plane with vertical coordinate $y$ and horizontal coordinate the parameter $\mu$ used to identify the different equations in the family. In detail, you are considering the family of differential equations

$$\frac{dy}{dt} = f(y, \mu)$$

where $\mu$ is considered to be constant in each equation. The bifurcation diagram is a picture in the $\mu, y$ plane that allows each line with constant $\mu$ to be seen as a phase line. Such a diagram must show the set where $f(y, \mu) = 0$ and something to determine the sign of this expression on the connected regions left when this curve is removed from the plane. At the points of this set where the partial derivative of $f(y, \mu)$ with respect to $y$ isn't zero, the phase line will have either a source or a sink. Furthermore, the implicit function theorem says that $f(y, \mu) = 0$ can be (locally) solved for $y$ as a function of $\mu$ at such points. The other points on the curve are special; they usually correspond to vertical tangents to the graph of $f(y, \mu) = 0$ and are called **bifurcation values**. Away from these points, the stationary points of the differential equation depend smoothly on $\mu$. At a typical bifurcation value, a pair of stationary points will appear in equations on one side of this value of $\mu$ with no corresponding stationary points for the equations on the other side. It is often easy to solve $f(y, \mu) = 0$ for $\mu$. These solutions may be used to graph this set, but you must remember to orient the graph with a horizontal $\mu$ axis.

Quiz #3 used

$$\frac{dy}{dt} = y^2 - 3y + a$$

(with parameter $a$ and variable $y$), and problem #3 on exam 1 used

$$\frac{dS}{dt} = 24S\left(1 - \frac{S}{8}\right) - \mu$$

(with parameter $\mu$ and variable $S$). Other examples appear in exercises 1–6 of section 1.7. Here are some additional exercises.

$$\frac{dy}{dt} = \mu + 2y - y^2$$

$$\frac{dy}{dt} = 24y\left(1 - \frac{y}{8}\right)\left(\frac{y}{3} - 1\right) - \mu$$

$$\frac{dy}{dt} = y^3 - \mu y^2 + 7y + 3$$

$$\frac{dy}{dt} = (y - \mu)\sin y$$

$$\frac{dy}{dt} = y^2 - \mu(y + 1)$$

**Chapter 3. Two dimensional Linear Homogeneous Systems.**

Chapter 2 introduced the use of systems, both directly and as an approach to higher order equations. However, the useful exam problems only appear when these ideas are developed in later chapters. In particular, the various types of two dimensional linear homogeneous systems are studied in chapter 3. The main result is that the solution of these equations are determined by combining **straight-line solutions** which are expressed in terms of the eigenvalues and eigenvectors of the coefficient matrix. This is most useful when the eigenvalues are real and distinct. Exercises 1–14 of Section 3.2 give many examples of this type. The origin may be classified as a **source, sink, of saddle** depending on the signs of the eigenvalues. This classification is an extensions of that used to describe phase lines in one dimension.

The same approach can be used when the eigenvalues are complex. However, algebra with complex numbers is less familiar, so the solution is cumbersome and mistakes are often made. The notes entitled “An easily remembered formula” from the Spring 2000 web page for this course describes an approach that finds the real solutions more reliably. The main result is that if $\mathbf{M}$ is a 2-by-2 matrix with eigenvalues $a \pm bi$, then $\mathbf{M}$ can be written as $a\mathbf{I} + b\mathbf{J}$ where $\mathbf{I}$ is the identity matrix and $\mathbf{J}$ is a matrix whose square is $-\mathbf{I}$. Once the eigenvalues of $\mathbf{M}$ are known, the equation $\mathbf{M} = a\mathbf{I} + b\mathbf{J}$ can be solved for $\mathbf{J}$. Then,

$$e^{\mathbf{M}t} = e^{at}\big(\mathbf{I}\cos bt + \mathbf{J}\sin bt\big).$$

The solution of $d\mathbf{Y}/dt = \mathbf{M}\mathbf{Y}$ with $\mathbf{Y}(0) = \mathbf{Y}_0$ is given by $\mathbf{Y} = e^{\mathbf{M}t}\mathbf{Y}_0$.

The origin is a **spiral** (if $a \neq 0$) or **center** (if $a = 0$) in this case. The spirals cases are further described as sources (if $a > 0$) or sinks (if $a < 0$). The direction of the spiral is either counterclockwise (**positive** in mathematics) or clockwise (**negative** in mathematics). This direction is given by the sign of the lower left entry in $\mathbf{M}$. Exercises 1–14 of Section 3.4 give practice with this type of equation.

**Special cases.** Section 3.5 introduces the unusual examples that arise when zero is an eigenvalue or when there is a repeated eigenvalue.

Repeated eigenvalues are expected to be special since there are usually not enough eigenvectors to diagonalize the matrix in this case. In the context of differential equations, this means that the straight-line solutions alone do not determine the general solution of the differential equation. An approach similar to the one used for complex eigenvalues applies can be used here to find the matrix exponential. If the only eigenvalue of $\mathbf{M}$ is $a$, then $\mathbf{M} = a\mathbf{I} + \mathbf{N}$. In this case $\mathbf{N}^2$ is the zero matrix, and

$$e^{\mathbf{N}}t = e^{at}\big(\,I + \mathbf{N}t\,\big).$$

There is no difficulty finding the solution in the case when zero is an eigenvalue. What is special in this case is that there is a **line of stationary points**. In the other cases, the origin is the only stationary point.

**Exercises.** On quizzes and exams, you will not be told what kind of eigenvalues to expect. The problems on quiz #4 and the first four problems on the second exam show what to expect.

Here are matrices from this year:

$$\begin{pmatrix} 1 & -1 \\ 3 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 \\ 5 & 5 \end{pmatrix} \qquad \begin{pmatrix} 6 & 3 \\ -4 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 11 & 14 \\ -7 & -3 \end{pmatrix} \qquad \begin{pmatrix} 3 & 2 \\ -2 & -2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix}$$

and here are some used in the past

$$\begin{pmatrix} 7 & 2 \\ 4 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 3 \\ -6 & -5 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 9 \\ 6 & 4 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -2 \\ 5 & 0 \end{pmatrix} \qquad \begin{pmatrix} -7 & -7 \\ 0 & 5 \end{pmatrix} \qquad \begin{pmatrix} -3 & 6 \\ 1 & -2 \end{pmatrix}$$

**Bifurcations for systems**. Section 3.7 extends the study of bifurcations to two dimensional systems. The setting is a linear homogeneous system, so the origin will be a stationary point. The classification of systems from the early part of the chapter identifies some types whose qualitative type will not change if the entries of the matrix are changed by a small amount. These are separated by some special types of systems. A **trace-determinant plane** is a convenient way to reduce the four matrix entries to two quantities that are essentially the coefficients of the characterisitc polynomial. The axes and the curve $T^2 = 4D$ divide this plane into regions that represent the main types of stationary point. The curves themselves give the special types. A family of linear systems depending on a parametergives a curve in this plane whose parametric equations are the expression of the coordinates in terms of the quantitiy parameterizing the system. Any time the curve crosses one of the axes or the curve $T^2 = 4D$, we say that there is a bifurcation.

Problem 5 on Exam 2 studied this for the matrix

$$\begin{pmatrix} a & 4 \\ -1 & 1 \end{pmatrix}$$

depending on a parameter $a$. Here, the trace is $a + 1$, the determinant is $a + 4$, and the **discriminant** $T^2 - 4D = a^2 - 2a - 15 = (a - 5)(a + 3)$. There is a bifurcation when any one of these quantities is zero. Thus there are bifurcations at $a = -4, -3, -1, and 5$. The intervals between these give the main types.

- When $a < -4$, the determinant is negative, so the origin is a **saddle**.
- When $-4 < a < -3$, the discriminant is positive, so the eigenvalues are real. The determinant is positive, so they have the same sign, and the trace is negative. This indicates that there are negative eigenvalues, and the origin is an **ordinary sink**.
- When $-3 < a < -1$, the discriminant has become negative while the trace remains negative. This indicates a **spiral sink**.
- When $-1 < a < 5$, the trace has become positive while the discriminant remains negative. This indicates a **spiral source**.
- When $5 < a$, trace, determinant, and discriminant are all positive. This indicates positive real eigenvalues, and the origin is an **ordinary source**.

The special values are:

- $a = -4$ has a zero eigenvalue and a line of stationary points.
- $a = -3$ is the special type with repeated eigenvalues. It is a sink for which the trajectories approaching the point *almost* spiral.
- $a = -1$ has **pure imaginary** eigenvalues. The stationary point is a **center**. The text treats this along with the spirals, but it is sufficiently special that it should be considered a bifurcation.
- $a = 5$ is another example of repeated eigenvalues.

Exercises 2–7 of Section 3.7 are typical.

**Chapter 4. Harmonic Motion, Forcing and Resonance.** Equations of the form

$$\frac{d^y}{dt^2} + p\frac{dy}{dt} + qy = g(t)$$

arise in the study of physical systems. Each term in this equation represent some kind of force. The numbers $p$ and $q$ are positive, but the function $g(t)$ representing the driving force will typically alternate between positive and negative values. The solution $y(t)$ gives the position of an object as a function of time. This one example is used to illustrate many important ideas:

1. A second order equation may be converted to a first order system in $(y, y')$. The derivative of $y$ is $y'$ and the derivative of $y'$ is $y''$, which is expressed in terms of $y$ and $y'$ by the given equation. This is important because the theory extends easily to systems.
2. The equation is solved when $y(t)$ is found since the second component in the vecto used to create a system is the derivative of the first. However, all theoretical properties are expressed using the system, so the initial conditions that should be specify to give a unique soution to the differential equation are the values of $y$ and $y'$ at one point.
3. The general solution of an inhomogenous linear equation is, as in the first order case, a sum of a **particular solution** and the general solution of the corresponding homogeneous equation. The **method of variation of parameters** can be used to obtain a particular solution as an integral, but it often easy to guess a finite dimensional space of functions that contains the solution. Finding the solution in this space is accomplished by solving an algebraic system of linear equations. (This is known as the **method of undetermined coefficients**).
4. In the equation of harmonic motion, if $g(x)$ is a linear combination of $\sin at$ and $\cos at$, the same will be true ot the solution $y(t)$, except when these functions satisfy the corresponding homogeneous equation. In the latter case, a particular solution can be found that is a linear combination of $t \sin at$ and $t \cos at$.

**Exercises**. Equations in this chapter of the text should give enough examples. Concentrate on general properties of the system rather than the special formulas used to explain **resonance** and **beats** in Section 4.3 or the nature of the **steady-state** solution derived in Section 4.4.

**Chapter 5. Qualitative Properties of Nonlinear Systems.** This chapter gives a systematic study of some properties introduced in Chapter 2. Systems that model **predator-prey** or **competing species** problems are good illustrations, because the stationary points are easy to find and lead to a simple dissection of the phase plane into regions where the solutions have fairly simple behavior. For these equations, the expressions for $dx/dt$ and $dy/dt$ are products of two linear expressions in $x$ and $y$. A slightly more complicated example was used on quiz #5 in which one of these factors was quadratic. All of these examples are **autonomous**, so that solutions give a family of well-defined disjoint **trajectories** in the phase plane.

A computer can show you a **slope field** in the pase plane, but this often leads to a cluttered picture with **too much information**. many qualitative properties of solutions need only know where $dx/dt$ and $dy/dt$ are positive and where they are negative. To find these regions, one first finds where these quantities are zero. That is, if the system is

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

the first step is to plot the **nullclines** $f(x, y) = 0$ and $g(x, y) = 0$. The **stationary points** are the points where **the different nullclines** meet. In our examples, $f(x, y)$ is a procuct of simple expressions. The places where each factor is zero is plotted separately, but the points of intersection of two parts of **the same** nullcline have no special significance. These curves are usually **not** trajectories. The equation $f(x, y) = 0$ identifies the points where the slope field is **vertical**; $g(x, y) = 0$ identifies the points where the slope field is **horizontal**. You don't need to memorize anything here, since this is just **common sense**. The expression $f(x, y)$ gives the horizontal component of the slope field: if it is zero, then there is only a vertical component. On the set where $g(x, y) = 0$, the sign of $f(x, y)$ indicates whethe the horizontal tangetnt points to the right ($f(x, y) > 0$, or left ($f(x, y) < 0$. On a region where $f(x, y)$ is defined and continuous (which we expect if we are trying to realize it as the derivative of something), the sign changes only where $f(x, y) = 0$. Since we are drawing these horizontal arrows where $g(x, y) = 0$, the direction changes only where **both** $f(x, y) = 0$ and $g(x, y) = 0$, i.e., at stationary points. Each region that does not meet a nullcline has directions lying in a particular quadrant. If the region is bounded by nullclines, the direction of the slope field is known exactly. In rare cases, it will be **along** the boundary. More often, the direction can be considered as pointing **in** or **out** relative to the region. Trajectories within one of the regions can also start or end at a stationary point on the boundary of the region. If two adjacent sides of a region are nullclines with an outward orientation, the regions can be partitioned according to which edge is reached by a trajectory starting a point in the region. These parts have a common boundary that is a curve that approaches the vertex between those edges. Such a curve is called a **separatrix**. A separatrix **is** a trajectory, so it can only be found by solving the differential equation, although some properties are easily determined from the equation.

The case in which both $f(x, y)$ and $g(x, y)$ are products of linear factors should not require any hints. On quiz #5,an example was used in which one branch of a nullcline was a circle. To use this, the curves were drawn, and contrasting colors were used for $f(x, y) = 0$ and $g(x, y) = 0$.

**Exercises**. In Section 5.2, Exercises 5, 6, 7, and 13 have the form that should be done with hints.

**Chapter 8. Discrete dynamical systems**.

Iteration of functions can be investigated easily using a programmable calculator. It will be assumed that simple experiments can be performed.

To define an iteration, start with a function $g(x)$ and some number $x_0$ and for $n \geq 0$ define $x_{n+1} = g(x_n)$.

If the iteration converges, the limit will be a **fixed point** of $g(x)$, i.e., a solution of $x = g(x)$. This allows an algebraic or graphical characterization of the fixed point.

Even if there is a fixed point, iteration may not converge to it. If $x_\infty$ is a fixed point of $g(x)$ for which $\left|g'(x)\right| < 1$, then there will be an interval $I$ around $x_\infty$ such that for all $x_0 \in I$, $\lim_{n\to\infty} x_n = x_\infty$.

If $\left|g'(x)\right| > 1$, the only iteration that will converge to $x_\infty$ is the constant sequence obtatined by taking $x_0 = x_\infty$.

Some iterations will approach periodic behavior. Others will appear to be random, or **chaotic**. There is a rich theory, including a **bifurcation theory** that has many common properties with other types of bifurcation met in the course.

There was only enough time for a brief introduction to this material, but questions involving a description of an iteration, as in Section 8.1, or classification of fixed points $x_\infty$ by the value of $g'(x_\infty)$ as in Section 8.2 are suitable for the exam.

**Exercises.** Since our treatment aims for bifurcation theory (without actually reaching it), the functions to experiment with contain a parameter. In each experiment, the parameter should be given a definite value. Functions that are easily programmed on a calculator are: $ax(1 - x)$; $\cos ax$; $\sin ax$. With trigonometric functions, it is important that you set your calculator in radian mode. A variant on this that is also easy to program adds a constant to the function. For accurate results, it is important that you **program** the function so it can be evaluated with a single press of a button. For exercises here, it sill suffice to have the function take its input from the display and return its result to the display.