SOME COMMENTS ON BIFURCATION

This is a slightly modified version of the notes posted on the same subject posted on the Math 252 web page and borrowed from the UTEP "SOS math" project. If you have time, you may want to look at their general differential-equations resources.

1. *Finding Bifurcation—an Analytic Approach.*

As we explained in class (and as the text explains too briefly, on p. 99), here is a quick way to seek bifurcation values of the parameter for a one-dimensional differential equation

$$\frac{dy}{dt} = f(y, \mu)$$

where $f(y, \mu)$ is a function for which both f and $\dfrac{\partial f}{\partial y}$ are continuous.

1. Solve the simultaneous equations

   $$\text{EQUI}: \qquad f(y, \mu) = 0$$
   $$\text{INDEF}: \qquad \frac{\partial f}{\partial y}(y, \mu) = 0$$

   for (what we hope are) a discrete set of pairs (y_k, μ_k). Let us call the solutions **critical pairs** of dependent-variable values and parameter values.

2. For each critical pair (y_0, μ_0), examine the behavior of the equation $\dfrac{dy}{dt} = f(y(t), \mu_0)$ near the equilibrium point $y = y_0$ and try to determine whether bifurcation in fact occurs.

The basic idea is that for a critical pair (y_0, μ_0), the condition EQUI makes y_0 an equilibrium point of the equation $y'(t) = f(y(t), \mu_0)$ while the condition $\dfrac{\partial f}{\partial y}(y_0, \mu_0) \neq 0$ would make one of the two conditions $f'(y_0, \mu) > 0$ *or* $f'(y_0, \mu) < 0$ hold for μ near μ_0 (by continuity) and thus "sources would remain sources and sinks would remain sinks" as we varied μ near μ_0. It follows that bifurcation can occur *only if* the INDEF condition $\dfrac{\partial f}{\partial y}(y_0, \mu_0) = 0$ holds, and so we cut down the set of μ's we have to examine for bifurcation to those μ's that belong to critical pairs. This procedure is very similar to the Calc I procedure in which, in order to maximize a function, you first found the critical points and then examined each critical point to find whether it was a maximizing point or not. (Examination was necessary because of functions like $f(x) = x^3$, for which $x = 0$ is a critical point but is neither a local maximum nor a local minimum.)

2. *A Detailed Example.*

This is the example of populations under constant fishing given on pp. 101–103 of the text. The equation is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - C$$

where we consider C to be the parameter "μ" whose bifurcation behavior we wish to study. (From the standpoint of the model: C is subject to easy human intervention, while k and N—even though they *could* be thought of as parameters—are given by the reproductive behavior of the fish and the carrying capacity of the environment and are thus difficult to control.) Studying just one parameter keeps things simple.

Applying the technique of §1 above—with "y" being P and "μ" being C—we set up the two equations EQUI and INDEF in `Maple`:

```
f := k*P*(1 - P/N)-C ;
fp := diff(f,P) ;
```

and find that the derivative is

$$k\left(1-\frac{P}{N}\right)-\frac{kP}{N}$$

so we solve:

```
solve({f=0, fp = 0},(P,C)) ;
```

getting exactly one critical pair:

$$\left\{P=\frac{1}{2}N,\ C=\frac{1}{4}kN\right\}\ .$$

So is $C_0 = \dfrac{kN}{4}$ a bifurcation value of the parameter? One must investigate. Fortunately, the quadratic character of $f(P,C)$ as a function of P lets us find its equilibrium points explicitly:

```
solve(f,P);
```

gives

$$\left\{\frac{kN+\sqrt{D}}{2k},\ \frac{kN-\sqrt{D}}{2k}\right\}$$

where $D = k^2N^2 - 4kCN$, so there are no (real!) equilibrium points if $D < 0$, *i.e.*, if $C > C_0$; there is exactly one equilibrium if $C = C_0$, and there are two equilibria if $C < C_0$. Thus **yes**, C_0 is a bifurcation value of the parameter C: "the number or the type of equilibria changes."

One could then go on and determine the types of the equilibria we have when $C < C_0$ and at $C = C_0$, as is done in the book.

Is this an easier method than the one in the book? It depends on what "easier" means. For the one-dimensional problems we are studying, the answer to the question depends on the problem. If the equations for the equilibrium are hard to solve, then having the second equation may help a lot, because the simultaneous solutions of two equations in two unknowns usually form a discrete set—frequently a finite set—and one may be able easily to eliminate one of the unknowns between the two equations. To take an example Mr Sontag made up, if

$$f(y,\mu) = \sin y - \mu\cos y$$

there are many possible equilibria for any given value of μ. (For example, when $\mu = 0$ and $f(y,0) = \sin y$, all the integer multiples of π are equilibria.) However, if we consider the simultaneous equations

$$\begin{aligned}\sin y - \mu\cos y &= 0\\ \cos y + \mu\sin y &= 0\end{aligned}$$

we can eliminate $\cos y$ to get

$$(1+\mu^2)\cdot\sin y = 0$$

but this can hold only if y is an integer multiple of π, and for these the simultaneous equations become

$$\begin{aligned}\mu\cos y &= 0\\ \cos y &= 0\end{aligned}$$

and the second of these is *never* satisfied for $y = n\pi$. So we know that there are *no bifurcations* and have done very little work. (In fact, though, we could have remembered that a linear combination of a sine and a cosine always looks like a translated sine:

$$\sin y - \mu\cos y = \sqrt{1+\mu^2}\cdot\sin(y-\varphi),\ \text{where } \varphi = \sin^{-1}\left(\frac{\mu}{\sqrt{1+\mu^2}}\right)$$

and it is clear from the graph of such a function that the derivative is nonzero at each root.) For higher-dimensional problems, this method *does* give a good systematic procedure, and one will use it if one can.

3. *Some Links.*

The Math 252 web page offers links to the following general references, and also to a couple of scientific papers (in web-readable form) that describe bifurcation behavior in some applications:

- Lecture notes on chaos and bifurcations (more for discrete systems, but it's the same idea). Has very nice pictures.
- A page on chaos.
- Chaos theory: a brief introduction.
- Chaos demonstrations from CalTech.
- Chaos-and-fractals web side.
- The Chaos Game by Devaney (one of the text authors).
- [Still] More Chaos notes.
- Chaos metalink.
- A picture of the bifurcation behavior for a two-parameter fishing model. Its accompanying narrative goes:

  A population model for haddock (Melanogrammus aeglefinus L.) developed by Horwood (Phil. Trans. R. Soc. Lond. B 350, 1995) is analysed further with respect to its ecological stability. It is shown that the dynamic properties are influenced primarily by zoöplankton production and harvesting intensity. The derived results relating to ecological stability are compared with available information from the North Sea and the George's Bank ecosystems. For a wide range of realistic parameter values, the predicted dynamics are characterised by fixed point dynamics; then the population is primarily destabilised by overfishing. High zoöplankton production, caused by either trends or fluctuations in production, may, however, drive the population into a region characterised by periodic fluctuations of varying fixed periods, and even aperiodic dynamics of closed curves and chaos. It is argued that assumed increased climactic variability may change the stability properties of the ecological system.

- Bifurcation in nonlinear convection, by Dr A. M. Rucklidge. Abstract:

  The aim of this course is to develop the theory of bifurcations in dissipative nonlinear systems, and to show how these techniques can be applied to specific physical problems. Convection in a fluid layer heated from below is a classic example of a system in which successive bifurcations lead from a trivial static state through ordered behaviour to disorder; moreover, the theory has important astrophysical and geophysical applications.

- Injection-induced bifurcations of transverse spatio-temporal patterns in semiconductor laser arrays by D. Merbach, O. Hess, H. Herzel, and E. Scholl. Abstract:

  We present results of numerical investigations on the complex spatio-temporal dynamics of semiconductor laser arrays. The diffusion of charge carriers turns out to be essential for instabilities in the output intensity above the laser threshold. Besides other bifurcations, a period doubling of a torus is found. The Karhunen-Loeve decomposition gives the dominant modes of the spatio-temporal dynamics of the output intensity and provides a measure of the number of spatio-temporal degrees of freedom.

- Analysis and modeling of bipedal gait dynamics:

  The main focus of the present investigation is the development of quantitative measures to assess the dynamic stability of human locomotion ... ... accomodates the study of the complex dynamics of human locomotion and differences among various individuals ... Changes in the stability of the biped as a result of bifurcations in the four-parameter parameter space are investigated.

- Bifurcation notes by Devaney (one of the textbook authors).