Math 291 Spring 2006: Formulas for the Final Exam

Distance between (x_0, y_0, z_0) and (x_1, y_1, z_1) is: $\sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}$.

Distance between (x_1, y_1, z_1) and $ax + by + cz + d = 0$ is: $\dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$.

A line with direction $\mathbf{v}$ through a point $\mathbf{r}_0$ is given by: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$.

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta) \quad \text{and} \quad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta)$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}, \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}, \quad \text{and} \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

The area of a parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a} \times \mathbf{b}|$.
The volume of a parallelepiped spanned by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

$$\text{comp}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \quad \text{and} \quad \text{proj}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\mathbf{a}$$

The linearization of (equation of the tangent plane to) $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$:

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

If $z = f(x_1, x_2, \ldots, x_n)$, then $dz = f_{x_1} dx_1 + f_{x_2} dx_2 + \cdots + f_{x_n} dx_n$.
The tangent plane to $F(x, y, z) = K$ at (a, b, c) is: $\nabla F(a, b, c) \cdot \langle x - a, y - b, z - c \rangle = 0$.

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \text{ and } D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}.$$

If $y = f(x)$ is given implicitly by $F(x, y) = K$, then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$.

If $z = f(x, y)$ is given implicitly by $F(x, y, z) = K$, then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$.

Arc length from a to t is given by the function $s(t) = \displaystyle\int_a^t |\mathbf{r}'(u)|\, du$. Thus $s'(t) = |\mathbf{r}'(t)|$.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\mathbf{T}'(t) = \kappa(t) s'(t) \mathbf{N}(t) \quad \mathbf{N}'(t) = -\kappa(t) s'(t) \mathbf{T}(t) + \tau(t) s'(t) \mathbf{B}(t) \quad \mathbf{B}'(t) = -\tau(t) s'(t) \mathbf{N}(t)$$

$$\kappa(t) = \frac{d\mathbf{T}}{ds} = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \qquad \text{For the curve } y = f(x)\text{: } \kappa(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}.$$

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} \qquad a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} \qquad \tau(t) = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}$$

For $z = f(x, y)$, define: $D(a, b) = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix} = f_{xx}(a, b) f_{yy}(a, b) - f_{xy}(a, b)^2$.

Let (a, b) be a critical point for $z = f(a, b)$. Then,

- If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $z = f(x, y)$ has a relative minimum at (a, b).
- If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $z = f(x, y)$ has a relative maximum at (a, b).
- If $D(a, b) < 0$, then $z = f(x, y)$ has a saddle point at (a, b).

Suppose that a large object of mass M is located at the origin and an object of mass m is located at the point $\langle x, y, z \rangle$. Then the gravatational force of M on m is $\mathbf{F}(x, y, z) = -\frac{mMG}{\sqrt{x^2+y^2+z^2}} \langle x, y, z \rangle$ where G is the gravatational constant.

Center of mass - 3 dimensions

$$\text{mass} = m = \iiint_E \rho(x,y,z)\,dV \qquad\qquad (\bar{x},\bar{y},\bar{z}) = \frac{1}{m}(M_{yz}, M_{xz}, M_{xy})$$

$$M_{yz} = \iiint_E x\,\rho(x,y,z)\,dV, \qquad M_{xz} = \iiint_E y\,\rho(x,y,z)\,dV, \qquad M_{xy} = \iiint_E z\,\rho(x,y,z)\,dV$$

Center of mass - a wire in the plane $\qquad (\bar{x},\bar{y}) = \frac{1}{m}(M_y, M_x)$

$$\text{mass} = m = \int_C \rho(x,y)\,ds, \qquad M_y = \int_C x\,\rho(x,y)\,ds, \qquad M_x = \int_C y\,\rho(x,y)\,ds$$

Polar: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $dA = r\,drd\theta$

Cylindrical: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$, $dV = r\,drd\theta dz$

Spherical: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$, $dV = \rho^2\sin(\phi)\,d\rho d\theta d\phi$

Change of variables: $$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv$$

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_S f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\,du\,dv\,dw$$

$$ds = |\mathbf{r}'(t)|dt \qquad dx = x'(t)dt \qquad \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$$

Green's Theorem: $$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = \iint_D \operatorname{curl}(\mathbf{F})\cdot\mathbf{k}\,dA$$

Surface Area: $$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA \qquad A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

Surface Integrals: $$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA$$

$$\mathbf{n} = \frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}, \qquad d\mathbf{S} = (\mathbf{r}_u\times\mathbf{r}_v)\,dA, \qquad dS = |\mathbf{r}_u\times\mathbf{r}_v|\,dA$$

If the surface is the graph of $z = g(x,y)$ then, $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)dA$

Stokes': $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$ $\qquad$ Divergence: $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV$

If $\mathbf{a} = \langle a_1, a_2, \dots, a_n\rangle$, then $\mathrm{dx}_j(\mathbf{a}) = a_j$.

$$\mathrm{dx}_{i_1}\wedge\mathrm{dx}_{i_2}\wedge\cdots\wedge\mathrm{dx}_{i_k}(\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_k) = \det\begin{pmatrix} \mathrm{dx}_{i_1}(\mathbf{a}_1) & \mathrm{dx}_{i_1}(\mathbf{a}_2) & \cdots & \mathrm{dx}_{i_1}(\mathbf{a}_k) \\ \mathrm{dx}_{i_2}(\mathbf{a}_1) & \mathrm{dx}_{i_2}(\mathbf{a}_2) & \cdots & \mathrm{dx}_{i_2}(\mathbf{a}_k) \\ \vdots & \vdots & & \vdots \\ \mathrm{dx}_{i_k}(\mathbf{a}_1) & \mathrm{dx}_{i_k}(\mathbf{a}_2) & \cdots & \mathrm{dx}_{i_k}(\mathbf{a}_k) \end{pmatrix}$$

Exterior Derivative:

If f is a 0-form, then $df = \frac{\partial f}{\partial x_1}dx_1 + \cdots \frac{\partial f}{\partial x_n}dx_n$.

If $\omega = \sum f_{i_1\dots i_k}dx_{i_1}\wedge\cdots\wedge dx_{i_k}$ is a k-form, then $d\omega = \sum(df_{i_1\dots i_k})\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}$.

Wedge Product:

If $\omega = \sum f_{i_1\dots i_k}dx_{i_1}\wedge\cdots\wedge dx_{i_k}$ is a k-form and $\eta = \sum g_{j_1\dots j_l}dx_{j_1}\wedge\cdots\wedge dx_{j_l}$ is an l-form, then $\omega\wedge\eta = \sum\sum f_{i_1\dots i_k}g_{j_1\dots j_l}dx_{i_1}\wedge\cdots\wedge dx_{i_k}\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_l}$ is a $(k+l)$-form.

Gen. Stokes' Theorem: $\int_M d\omega = \int_{\partial M}\omega$ where M is a k-manifold and ω is a $(k-1)$-form.