Mathematics 244 Essay 2

Matrix Exponentials

Fall 2004

0. Introduction and Setup

The general solution of the first-order, linear, constant-coefficient system

$$\frac{dY}{dt} = MY, \tag{1}$$

where M is an n by n matrix, can be described in terms of a matrix exponential e^{Mt}.

A useful characterization is that e^{Mt} is the unique solution of $dY/dt = MY$ with $Y(0) = I$. The **existence and uniqueness theorem** applies to systems, so once you know that you have a solution to an initial value problem, it must be the only solution, and the method that finds it can be considered valid.

This essay shows examples of the use of this characterization. Although the method applies more generally, the cases illustrated here will serve as models for all examples in this course. Details have been skipped, but the results are easily checked.

1. Eigenvectors and Projections

The textbook chooses to **define** the matrix exponential by a formula involving the eigenvalues and eigenvectors of M. This works reasonably well if $n = 2$ and the matrix M has integer entries and a characteristic polynomial that factors to reveal distinct integer eigenvalues. The determination of eigenvalues is easy in this case.

Example 1

Let

$$M = \begin{bmatrix} 5 & -4 \\ 3 & -3 \end{bmatrix}$$

A **formula for the characteristic polynomial** of a 2 by 2 matrix can be found by expanding $\det(M - \lambda I)$ for a matrix M all of whose entries are distinct variables. The result is that the characteristic polynomial in this case is $\lambda^2 - (\operatorname{tr} M)\lambda + (\det M)$, where $\operatorname{tr} M$ denotes the **trace** of the matrix which is the **sum of the diagonal elements**. Because these are coefficients of the characteristic polynomial, the trace is also the **sum of the roots** and the determinant is the **product of the roots**.

For this matrix, $\operatorname{tr} M = (5) + (-3) = 2$ and $\det M = (5)(-3) - (-4)(3) = -15 + 12 = -3$ so the characteristic polynomial is $\lambda^2 - 2\lambda - 3$. This is easily seen to have roots 3 and -1 corresponding to the factorization $(\lambda - 3)(\lambda + 1)$. These roots are the eigenvalues of M. To find the eigenvectors, subtract the eigenvalue from the diagonal of M and find a **non-zero column vector** such that the product of the modified matrix with the column is a **zero vector**. Such a vector must exist. **If you can't find one, then you have made a mistake.** Any nonzero multiple of an eigenvector is also an eigenvector, so you can choose one that is easiest to write. Also, note that subtracting the root is equivalent to forming the factor of the characteristic polynomial evaluated at the matrix M. Thus,

$$M - 3I = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix} \quad \text{and} \quad M + I = \begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix}$$

For each of these matrices, each row says the same thing about a column that it can multiply to give zero: for $M - 3I$, the entries in the column are proportional to $(2, 1)$; for $M + I$, the entries are proportional to $(2, 3)$. It is customary to say that these vectors are **the** eigenvectors, although any nonzero multiple would do just as well.

The general solution of the differential equation $dY/dt = MY$ is

$$Y = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-t}.$$

If you check this by substituting this expression in the equation, the role of the eigenvalues and eigenvectors will be clear. To get the matrix exponential, you need to find values of c_1 and c_2 that give a value of Y whose value at zero is the column whose entries are $(1, 0)$ and values of c_1 and c_2 that give a value of Y whose value at zero is the column whose entries are $(0, 1)$. The corresponding Y are the first and second columns, respectively, of e^{Mt}. These equations can be solved because **eigenvectors for different eigenvalues are always linearly independent**. These equations are easy enough to solve that we just give the answer, leaving details to the reader.

$$\begin{aligned} e^{Mt} &= \frac{1}{4} \begin{bmatrix} 6e^{3t} - 2e^{-t} & 4e^{-t} - 4e^{3t} \\ 3e^{3t} - 3e^{-t} & 6e^{-t} - 2e^{3t} \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix} e^{3t} + \frac{1}{4} \begin{bmatrix} -2 & 4 \\ -3 & 6 \end{bmatrix} e^{-t} \\ &= \frac{1}{4} \big((M + I)e^{3t} - (M - 3I)e^{-t} \big) \end{aligned}$$

This suggests that e^{Mt} can be found without all of the computation suggested by our definition. It also reveals that, not only are the columns of the factors $(M + I)$ and $(M - 3I)$ eigenvectors for **the other** factor, but they can be scaled to be exactly what is need for e^{Mt}. The nature of the scale factor will be considered after the second example.

Example 2 The computation becomes more difficult for larger matrices, but there is one case that is fairly easy. Consider

$$M = \begin{bmatrix} 3 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Such a matrix is called **triangular** because the entries on one side of the **main diagonal** are zero. This diagonal from upper left to lower right plays a special role: it gives the locations of the 1's in the identity matrix, so it coincides with the entries modified in calculating the characteristic polynomial. Since the determinant of a triangular matrix is the product of the diagonal entries, it follows that the eigenvalues of a triangular matrix are just the diagonal entries. In this example, the eigenvalues of M are 3, 1 and -1. If these are all different, each eigenvalue will lead to a different eigenvector. Sums of these vectors, multiplied by arbitrary constants and the functions $e^{\lambda t}$, are solutions of $dY/dt = MY$. General principles guarantee that these eigenvectors are linearly independent, so values of the constants can be found to satisfy any initial conditions.

The eigenvectors are the columns whose products with the $M - \lambda I$ are zero, so the first step in finding e^{Mt} is to construct

$$A = M - 3I = \begin{bmatrix} 0 & 2 & 4 \\ 0 & -2 & 1 \\ 0 & 0 & -4 \end{bmatrix} \qquad B = M - I = \begin{bmatrix} 2 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \qquad C = M + I = \begin{bmatrix} 4 & 2 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

By solving systems of equations, the entries of the eigenvectors are seen to be proportional to $(1, 0, 0)$, $(1, -1, 0)$, and $(3, 2, -4)$. These are the columns in the matrix of coefficients of the equations that need to be solved to find the coefficients in the general solution that match given initial conditions.

The work needed to do this is not difficult, but it takes long enough that errors may creep in. We can take advantage of the special form of the matrix to give a short method that can be seen to give e^{Mt} more directly and in a form that includes several checks that can be used to avoid mistakes. First, form

$$BC = \begin{bmatrix} 8 & 8 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad CA = \begin{bmatrix} 0 & 4 & 2 \\ 0 & -4 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad AB = \begin{bmatrix} 0 & 0 & -6 \\ 0 & 0 & -4 \\ 0 & 0 & 8 \end{bmatrix}$$

A general principle, called the **Cayley-Hamilton Theorem**, says that substituting a matrix into its characteristic polynomial always yields zero. Thus, the columns of these matrices will be eigenvectors for the eigenvalue that was **not** used in producing its factors. From this, it follows that the square of each these matrices is a constant multiple of itself. For example, the columns of BC are eigenvectors for the eigenvalue 3. Since M acts like multiplication by 3 on BC, $B = M - I$ acts like multiplication by $3 - 1 = 2$ and $C = M + I$ acts like multiplication by $3 + 1 = 4$. Thus $(BC)^2 = 8(BC)$.

The matrices $(1/8)BC$, $(-1/4)CA$ and $(1/8)AB$ act on the eigenvectors by sending two of them to zero and acting like the identity on the third. This can be used to separate the components in an expression of a general vector as a linear combination of eigenvectors. From this, it follows that

$$e^{Mt} = \frac{1}{8}BCe^{3t} - \frac{1}{4}CAe^{t} + \frac{1}{8}ABe^{-t}.$$

As with other techniques developed to solve differential equations, it is not necessary to have a complete proof of the formula to use it with confidence, since it is easy to check in any individual case that the derivative of the matrix is its product with M and the value at $t = 0$ is the identity matrix.

In Example 1, each $M - \lambda I$ already has a square that is a scalar multiple of itself. Dividing by these factors, which are $\pm(\lambda_1 - \lambda_2)$, gives the coefficients seen in that example.

The method of Example 2 works for triangular matrices of any size with no repetitions on the diagonal. Each diagonal entry is an eigenvalue, and for each eigenvalue λ, the coefficient E_λ of $e^{\lambda t}$ in the expression for e^{Mt} is a scalar multiple of the product of $(M - \mu I)$ for **all of the other** eigenvalues μ. There is even a shortcut for finding those multiples. Each E_λ has zero on the diagonal except in the position occupied by λ in M, so that position must contain the number 1. After multiplying the $(M - \mu I)$, it is only necessary to divide by the only nonzero quantity on the diagonal. As with any shortcut, the result should be checked by showing that the entries in the off-diagonal positions of all E_λ sum to zero. Although the computations in this method are immune to most common computational errors, they should not be accepted until they are checked. Fortunately, the check is also very easy.

2. Complex Eigenvalues

The direct method for construction e^{Mt} is even more impressive in the case of complex eigenvalues. First, look at the method described in the textbook, skipping details.

Suppose now that $n = 2$ and the eigenvalues of M are $r \pm si$ with $s \neq 0$. Then all entries of all solutions of $dY/dt = MY$ can be expressed in terms of $e^{(r\pm si)t}$. If the resulting expression is expanded and the complex exponentials $e^{(r\pm si)t}$ converted to real exponentials and trigonometric functions, then

$$e^{Mt} = Pe^{rt}\cos st + Qe^{rt}\sin st,$$

where P and Q are 2 by 2 matrices of real numbers. It turns out to be **much easier** to **identify** P and Q from this equation than to **calculate** them using eigenvectors. In particular, putting $t = 0$ in this expression leads (**immediately!**) to $P = I$.

To find Q, we can differentiate e^{Mt}. For the discovery of the solution, it suffices to consider only the value at $t = 0$. This should be M, and direct calculation shows it to be $rP + sQ$. Since we have $P = I$, knowing r and s allows us to obtain Q as $(1/s)(M - rI)$. The Cayley-Hamilton Theorem, which can be verified by direct computation for 2 by 2 matrices, shows that the square of Q is $-I$. A change to more suggestive notation gives

Theorem 1. *If $n = 2$ and M has eigenvalues $r \pm si$, then*

$$e^{Mt} = e^{rt}(I \cos st + J \sin st). \qquad (E)$$

where I is the identity and J is characterized by $M = rI + sJ$. Furthermore, $J^2 = -I$, so that it has trace zero, and $\operatorname{tr} M = 2r$. Here, the matrix J plays the role of the number i in the algebra of polynomials in M with real coefficients (all of which commute with M).

This shows that the e^{Mt} can be written as soon as we know the eigenvalues, but Theorem!1 shows that we can make the solution even simpler. The characteristic polynomial of a 2 by 2 matrix M can be written as

$$\lambda^2 - \operatorname{tr}(M)\lambda + \det(M)$$

If, after writing the characteristic polynomial, we discover that the eigenvalues are complex, the roots $r + si$ are usually found by **completing the square**. Then, J is found by subtracting a rI from M and dividing by s. However, we have noted that r is $\operatorname{tr}(M)/2$, so we could **begin this process without solving the equation**. Since the trace is a **linear function of matrices**, the matrix $M - rI$ has zero trace. For 2 by 2 matrices, this means that the diagonal entries are negatives of one another. Hence, one should pause after finding it to check that one has chosen the correct sign for r. Since $\det(J) = 1$, we must have $\det(M - rI) = s^2$. For at least the matrices appearing in our exercises, it will be easier to find $\det(M - rI)$ than $\det(M)$. Moreover, the former is s^2 while the latter needs **further processing** by the **quadratic formula** or the **completing the square** process.

Example 3

Let

$$M = \begin{pmatrix} 19 & -39 \\ 6 & -11 \end{pmatrix}.$$

We have $\operatorname{tr}(M) = 8$, so we form

$$M - 4I = \begin{pmatrix} 15 & -39 \\ 6 & -15 \end{pmatrix}$$

to get a matrix of trace zero. It is easily seen that this has determinant 9, so we multiply by $1/3$ to get

$$J = \begin{pmatrix} 5 & -13 \\ 2 & -5 \end{pmatrix}$$

and $M = 4I + 3J$. Thus, the eigenvalues are $4 \pm 3i$ and

$$e^{Mt} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{4t} \cos 3t + \begin{pmatrix} 5 & -13 \\ 2 & -5 \end{pmatrix} e^{2t} \sin 3t.$$

If there is any doubt, a straightforward evaluation of its derivative would show that this is correct. Note that the initial conditions are clearly satisfied.

General properties

The key to finding e^{Mt} in Example 3 was to consider the relation between the exponentials of all matrices in $\mathbb{R}[M]$, to select J as the matrix whose exponential was easiest to find. The validity of this approach is based on the following result that we state without proof.

Lemma. *If $NM = MN$, then $Ne^{Mt} = e^{Mt}N$. Also, $Ne^{Mt} = e^{Mt}N$ implies that $NM = MN$.*

Corollary. *If $AB = BA$, then $e^{(A+B)t} = e^{At}e^{Bt}$*

Proof. The **product rule** of calculus holds for matrix products provided that the order of factors is respected. Thus

$$\frac{d}{dt}\left(e^{At}e^{Bt}\right) = (Ae^{At})e^{Bt} + e^{At}(Be^{Bt}).$$

The lemma allows this to be written as

$$Ae^{At}e^{Bt} + Be^{At}e^{Bt} = (A+B)e^{At}e^{Bt}.$$

Thus, $e^{(A+B)t}$ and $e^{At}e^{Bt}$ have been shown to satisfy the same initial value problem, which requires that they be equal.

In Example 3, A was rI, so e^{At} was $e^{rt}I$ and B was sJ, so e^{Bt} was $I\cos st + J\sin st$. The derivative of this expression is $-sI\sin st + sJ\cos st = -sJ^2\sin st + sJ\cos st$. Thus, differentiation multiplies this expression by sJ.

More two by two matrices

For any 2 by 2 matrix M, if $M = rI + Q$, then $e^{Mt} = e^{rt}e^{Qt}$. We have seen that matrices Q of trace zero play a special role, and r can be chosen to reduce to this special case. Since $e^{Bt} = I\cos t + B\sin t$ when $\det B = 1$, we can **guess that** $e^{Bt} = I\cosh t + B\sinh t$ when $\det B = -1$, and this is easily verified. As before, other negative determinants are covered by taking a suitable constant multiple of t in this expression.

Although we began with a solution using **projections** when there are two distinct real eigenvalues, the solution using **hyperbolic functions** fits into our general pattern, and the exponentials are easily recovered from the formulas $\cosh st = (e^{st} + e^{-st})/2$ and $\sinh st = (e^{st} - e^{-st})/2$.

The definition of e^{Bt} as a series also shows that $e^{Bt} = I + Bt$ when $\det B = 0$. Again, however one **guesses this solution**, a proof consists of showing that it satisfies (1) and reduces to I when $t = 0$. This use of **leading special cases** seems much more robust than the traditional solution. Having derived the special case **once**, they are available for as long as you remember them (which should be a long time). Extension to other cases uses only the **addition formula** for the exponential. Although this formula requires that the matrices appearing in it commute, this property holds for the algebra of all polynomials in a single matrix.

Since there are formulas covering all matrices of trace zero, the reduction to this case can be done before learning which case will apply.

Example 1 revisited

In example 1, we had

$$M = \begin{bmatrix} 5 & -4 \\ 3 & -3 \end{bmatrix}$$

Here, $\mathrm{tr}(M) = 2$, so we take $r = 1$ and

$$M - rI = \begin{bmatrix} 4 & -4 \\ 3 & -4 \end{bmatrix}$$

which has determinant -4. The square root of the absolute value of this is 2, so we set $s = 2$ and multiply by $1/2$ to get the coefficient of $\sinh st$. That is, $M = I + 2H$, where

$$H = \begin{bmatrix} 2 & -2 \\ 3/2 & -2 \end{bmatrix}$$

has trace zero and determinant -1, so that $H^2 = I$. This can be done for any 2 by 2 matrix, and we have

Theorem 2. *If $n = 2$ and M has eigenvalues $r \pm s$, then*

$$e^{Mt} = e^{rt}(I \cosh st + H \sinh st). \qquad (H)$$

where I is the identity and H is characterized by $M = rI + sH$. Furthermore, $H^2 = I$, so that it has trace zero, and $\operatorname{tr} M = 2r$. Formula (H) can be converted to **constant** *linear combinations of $e^{(1+2)t} = e^{3t}$ and $e^{(1-2)t} = e^{-t}$ by converting the hyperbolic functions to their exponential equivalents. In terms of the matrices in this expression, we get*

$$e^{Mt} = \frac{I+H}{2} e^{(r+s)t} + \frac{I-H}{2} e^{(r-s)t}. \qquad (H')$$

As in the trigonometric case, we can begin by finding $r = \operatorname{tr}(M)/2$. Then $M - rI$ needs to be written as sH. The definition of H requires that the determinant of sH be $-s^2$, so that s can be found from the matrix $M - rI$, and H is found by multiplying this matrix by $1/s$. In Example 1,

$$\begin{aligned} e^{Mt} &= e^{t}\left(I \cosh 2t + H \sinh 2t \right) \\ &= \frac{I+H}{2} e^{3t} + \frac{I-H}{2} e^{-t}. \end{aligned}$$

This In this case, this reduces to the expression that we met earlier. More generally, using $H = (1/s)(M - rI)$, we recover the projections met earlier:

$$e^{Mt} = \frac{M - (r-s)I}{2s} e^{(r+s)t} + \frac{(r+s)I - M}{2s} e^{(r-s)t}.$$

However, this has little to recommend it over (H') since we will have already found r, s and H, and the actual denominator of the entries of H may be smaller than s.

Example 4 Let

$$M = \begin{bmatrix} 9 & 4 \\ -9 & -3 \end{bmatrix}$$

Then, $\operatorname{tr} M = 9 - 3 = 6$, so we should write $M = 3I + N$ to introduce a matrix N of trace zero. Here,

$$N = \begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix}$$

In this case, $\det N = 0$, so $e^{Nt} = I + Nt$ and $e^{Mt} = e^{3t}(I + Nt)$. That is,

$$e^{Mt} = e^{3t}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix} t \right).$$

As in the other examples, this is easily checked. The general result is

Theorem 3. *If $n = 2$ and M has r as a double eigenvalue, then*

$$e^{Mt} = e^{rt}(I + Kt). \qquad (R)$$

where I is the identity and HK is characterized by $M = rI + K$. Furthermore, K^2 is the zero matrix, so that it has trace zero, and $\operatorname{tr} M = 2r$.

A general method

Combining Theorems 1, 2, and 3, there is a **single procedure** for finding e^{Mt} for all 2 by 2 matrices M:

1. define r by $2r = \operatorname{tr} M$;
2. form $M - rI$, compute its determinant, and branch to the appropriate case
   (a) if $\det(M - rI) > 0$, define s by $s^2 = \det(M - rI)$ and J by $M - rI = sJ$,
   (b) if $\det(M - rI) < 0$, define s by $s^2 = -\det(M - rI)$ and H by $M - rI = sH$,
   (c) if $\det(M - rI) = 0$, define K by $M - rI = K$;
3. write the answer $e^{Mt} = e^{rt}(I \cos t + J \sin t)$ or $e^{rt}(I \cosh t + H \sinh t)$ or $e^{rt}(I + Kt)$.

Initial vectors

If v is any **constant vector**, then $y = e^{Mt}v$ satisfies

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}\left(e^{Mt}v\right) \\ &= \left(\frac{d}{dt}e^{Mt}\right)v \\ &= \left(Me^{Mt}\right)v \\ &= M\left(e^{Mt}v\right) \\ &= My \end{aligned}$$

so it satisfies the equation $dy/dt = My$. Also, its value at $t = 0$ is just v. That is, once you have e^{Mt}, you need only multiply it by the **given initial conditions** to find the solution of $dy/dt = My$ with those initial values.

Exact solutions of initial value problems usually begin be finding a general solution and then determining the values of the parameters to meet the initial conditions. Such a general solution must include all of the **linearly independent columns** of e^{Mt}, so there is no saving in using a different form of the general solution. On the contrary, **any other form is wasteful** when it comes to solving initial value problems at $t = 0$ since there are no equations to solve once you have e^{Mt} — you only need to multiply this matrix by the given initial vector.

Exercises

Find e^{Mt} for the following matrices M. Then, solve $dY/dt = MY$ with the initial condition $Y(0) = v$.

(#1) $M = \begin{bmatrix} 23 & -30 \\ 15 & -19 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

(#2) $M = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$.

(#3) $M = \begin{bmatrix} 1 & 6 \\ -4 & -9 \end{bmatrix}$, $v = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$.

(#4) $M = \begin{bmatrix} -7 & 25 \\ -1 & 3 \end{bmatrix}$, $v = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$.

(#5) $M = \begin{bmatrix} 19 & -10 \\ 20 & -11 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(#6) $M = \begin{bmatrix} -16 & 27 \\ -12 & 20 \end{bmatrix}$, $v = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

(#7) $M = \begin{bmatrix} 23 & -12 \\ 30 & -13 \end{bmatrix}$, $v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

(#8) $M = \begin{bmatrix} 5 & 24 & 33 \\ 0 & -3 & -10 \\ 0 & 0 & 2 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.