# Math 321 — Fall 2006
# MW5 SEC-220
# Prof. Bumby

**Comments on fourth lecture 09/18/06**

Exercise 13.1

The discussion of this exercise in class was too brief. Here is a little more.

In the **overdamped** case, $c^2 > 4mk$, the motion is given by

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{13.1}$$

where $r_1 > r_2$ are both negative. In particular,

$$r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m} = \frac{-2k}{c + \sqrt{c^2 - 4mk}}$$

and $r_2$ is given by a similar expression with a minus sign in front of the radical sign.

The exercise asks about solving $x = 0$ for $t$. The solution of this equation is fairly simple:

$$\begin{aligned} c_1 e^{r_1 t} + c_2 e^{r_2 t} &= 0 \\ c_1 e^{r_1 t} &= -c_2 e^{r_2 t} \\ e^{(r_1 - r_2)t} &= -c_2/c_1 \\ (r_1 - r_2)t &= \ln(-c_2/c_1) \end{aligned}$$

This says that there is no real solution if $c_1$ and $c_2$ have the same sign and exactly one real solution if they have opposite signs. The solution is positive if $|c_1| > |c_2|$. The exercise asks to relate this to the initial values, $x(0)$ and $x'(0)$.

Before beginning, it should be noted (following section 6) that the exponentials are dimensionless quantities, so that the $c_i$ must have units of **distance** and the $1/r_i$ have units of **time**. In the case of the $r_i$, this follows from the units used to express $m$, $c$ and $k$, but the $c_i$ will be found from the initial conditions. We write $x(0) = x_0$ and $x'(0) = v_0$. There is no loss of generality in assuming $x_0 > 0$; but the sign of $v_0$ must be noted carefully. A positive sign identifies an initial motion away from the equilibrium position, while a negative sign means motion toward the equilibrium position (this qualitative information is really a property of the ratio $v_0/x_0$). Since $e^0 = 1$, $c_1 + c_2 = x_0$. The derivative of $x(t)$ is easily seen to be $c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}$, so $c_1 r_1 + c_2 r_2 = v_0$. Solving for the $c_i$ gives

$$\begin{aligned} c_1 &= \frac{r_1 x_0 - v_0}{r_1 - r_2} \\ c_2 &= \frac{v_0 - r_2 x_0}{r_1 - r_2} \end{aligned}$$

(solving the equations is easy, but you should note that these answers are **more** easily checked). From these expressions, it follows that the dimensions are correct since the numerators are velocities and the reciprocals of denominators are measurements of time, and that $c_1$ and $c_2$ have the same sign exactly when

$$r_1 x_0 > v_0 > r_2 x_0$$

(using our convention that $x_0 > 0$). Since both $r_i < 0$, the velocities leading to a simple decay for **all** time are negative. To determine whether a solution with another initial velocity crosses the equilibrium position in the past or in the future, we need to look at the relative sizes of $c_1$ and $c_2$. Fortunately, this is easy. If $v_0 > r_1 x_0$, then

$$v_0 - r_2 x_0 = v_0 - r_1 x_0 + (r_1 - r_2) x_0 > v_0 - r_1 x_0 > 0,$$

so $-c_2 > c_1 > 0$. In this case, the equilibrium crossing is in past. This case includes $v_0 > 0$.

Similarly, $v_0 < r_2 x_0$ leads to a future crossing of the equilibrium position.

You should do exercise 13.2 that asks for a similar analysis using the formulas for the **critically damped** case $c^2 > 4mk$.

With these results, and those for nearby **underdamped** oscillations, you should examine the claim at the top of page 41 that these distinctions are **physically insignificant** by noting that there will be either zero or one crossing of the equilibrium position in the time interval $0 < t < T$ for fixed $T$ for an interval of values of $c$ surrounding the **critical** value $c = 2\sqrt{mk}$.

End of discussion