MATH 336 CLASS NOTES OCT. 2, 2008

Two species model with competition

$$\frac{dN_1}{dt} = N_1\, g_1(N_1, N_2) = r_1 N_1 \left(1 - \frac{N_1 + \beta_{12} N_2}{K_1}\right), \quad g_1(N_1, N_2) = 1 - \frac{N_1 + \beta_{12} N_2}{K_1}$$

$$\frac{dN_2}{dt} = N_2\, g_2(N_1, N_2) = r_2 N_2 \left(1 - \frac{N_2 + \beta_{21} N_1}{K_2}\right), \quad g_2(N_1, N_2) = 1 - \frac{N_2 + \beta_{21} N_1}{K_2}$$

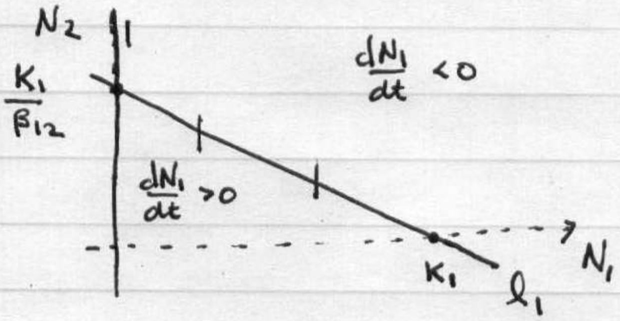
$$J(N_1, N_2) = \begin{bmatrix} \frac{\partial}{\partial N_1}\left(N_1 g_1(N_1, N_2)\right) & \frac{\partial}{\partial N_2}\left(N_1 g_1(N_1, N_2)\right) \\ \frac{\partial}{\partial N_1}\left(N_2 g_2(N_1, N_2)\right) & \frac{\partial}{\partial N_2}\left(N_2 g_2(N_1, N_2)\right) \end{bmatrix}$$

$$= \begin{bmatrix} g_1(N_1, N_2) - \frac{r_1}{K_1} N_1 & -\frac{r_1 \beta_{12}}{K_1} N_1 \\ -\frac{r_2 \beta_{21} N_2}{K_2} & g_2(N_1, N_2) - \frac{r_2}{K_2} N_2 \end{bmatrix}$$

<u>N_1 clines:</u> The N_1-cline consists of the line $N_1 = 0$ and the graph of $0 = g_1(N_1, N_2) = 1 - \frac{N_1 + \beta_{12} N_2}{K_1}$, which is the line $N_2 = \frac{K_1}{\beta_{12}} - \frac{N_1}{\beta_{12}}$

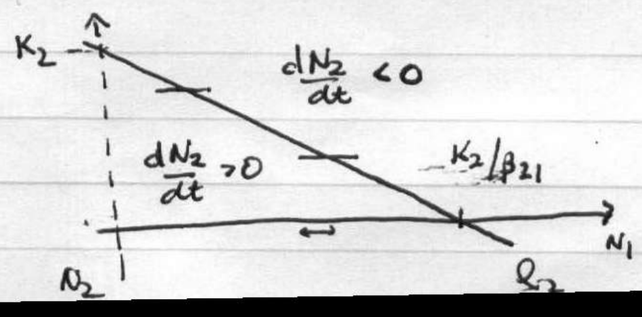
Call this line ℓ_1

Note: When (N_1, N_2) is above ℓ_1 in the 1st quadrant $\frac{dN_1}{dt} < 0$; when it is below $\frac{dN_1}{dt} > 0$



<u>N_2-cline.</u> This consists of $\{N_2 = 0\}$ and the line ℓ_2, which is $N_2 = K_2 - \beta_{21} N_1$.

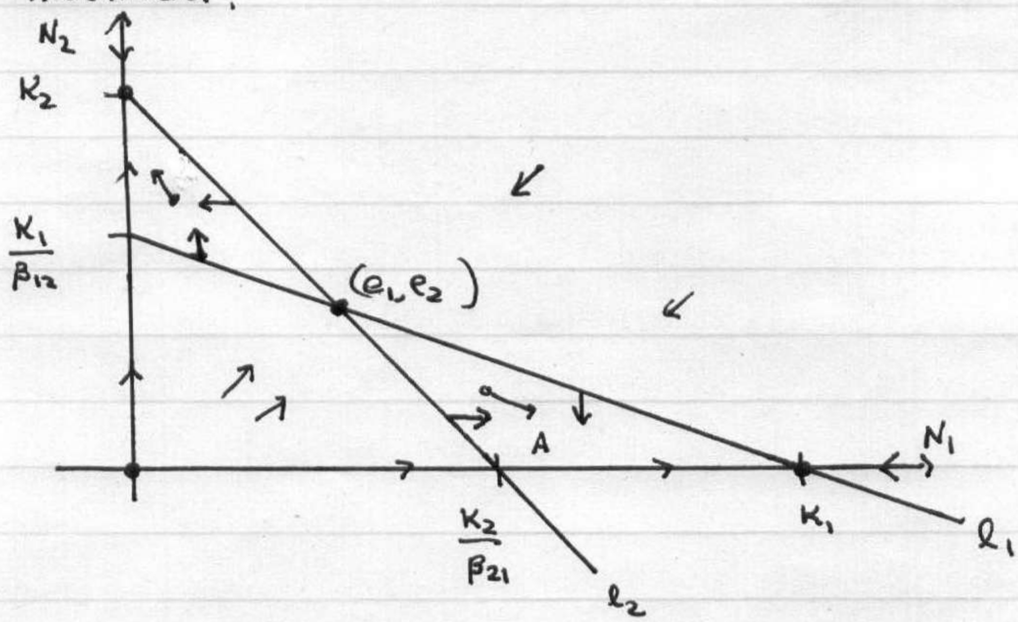
The figure also shows the sign of $\frac{dN_2}{dt}$ above and below ℓ_2 in the 1st quadrant.



We shall consider the case $K_2 > \frac{K_1}{\beta_{12}}$ $K_1 > \frac{K_2}{\beta_{21}}$.

[This case requires $K_2 > \frac{K_1}{\beta_{12}} > \frac{K_2}{\beta_{12}\beta_{21}}$, and hence requires $\beta_{12}\beta_{21} > 1$]

The null clines and some directions of the direction field are illustrated.



<u>Observe</u>: 1) When $N_2(0)=0$ $N_2(t)=0$ for all t, and $N_1(t)$ satisfies the logistic equation $\frac{dN_1}{dt} = r_1 N_1 (1 - \frac{N_1}{K_1})$. If $N_1(0) > 0$ then $\lim_{t\to\infty} N_1(t) = K_1$. This is illustrated by the arrows on the N_1-axis.

Consider a solution starting at a point $(N_1(0), N_2(0))$ in the region labeled A in the diagram. The solution curve must move in a south easterly direction. It cannot cross any of the boundaries because the direction field along the boundaries of region A defined by ℓ_1 and ℓ_2 point <u>into</u> A. It cannot exit through the N_1 axis, because solution curves do not intersect and the N_1-axis is a solution curve. Let $(N_1(t), N_2(t))$ denote the solution. It is clear that $\lim_{t\to\infty} (N_1(t), N_2(t))$ must exist. But we know any such limit must be an equilibrium point. Since $(K_1, 0)$, is an equilibrium point and the only other equilibrium point bounding A is at (e_1, e_2), we see

$$\lim_{t\to\infty} (N_1(t), N_2(t)) = (K_1, 0)$$

for all solutions starting in A.

By studying the direction field near $(K_1, 0)$ we see that it must be stable. We can confirm this and obtain an accurate picture of the phase portrait near $(K_1, 0)$ by studying the Jacobian matrix $J(K_1, 0)$. This is

$$J(K_1, 0) = \begin{bmatrix} -r_1 & -r_1\beta_{12} \\ 0 & r_2\left(1 - \frac{\beta_{21}}{K_2}K_1\right) \end{bmatrix}$$

$$= \begin{bmatrix} -r_1 & -r_1\beta_{12} \\ 0 & -\delta \end{bmatrix}$$

where $\delta = r_2\left(K_1\frac{\beta_{21}}{K_2} - 1\right)$. Since $K_1\frac{\beta_{21}}{K_2} > 1$ by assumption, $J(K_1, 0)$ has two negative eigenvalues and hence $(K_1, 0)$ is stable.

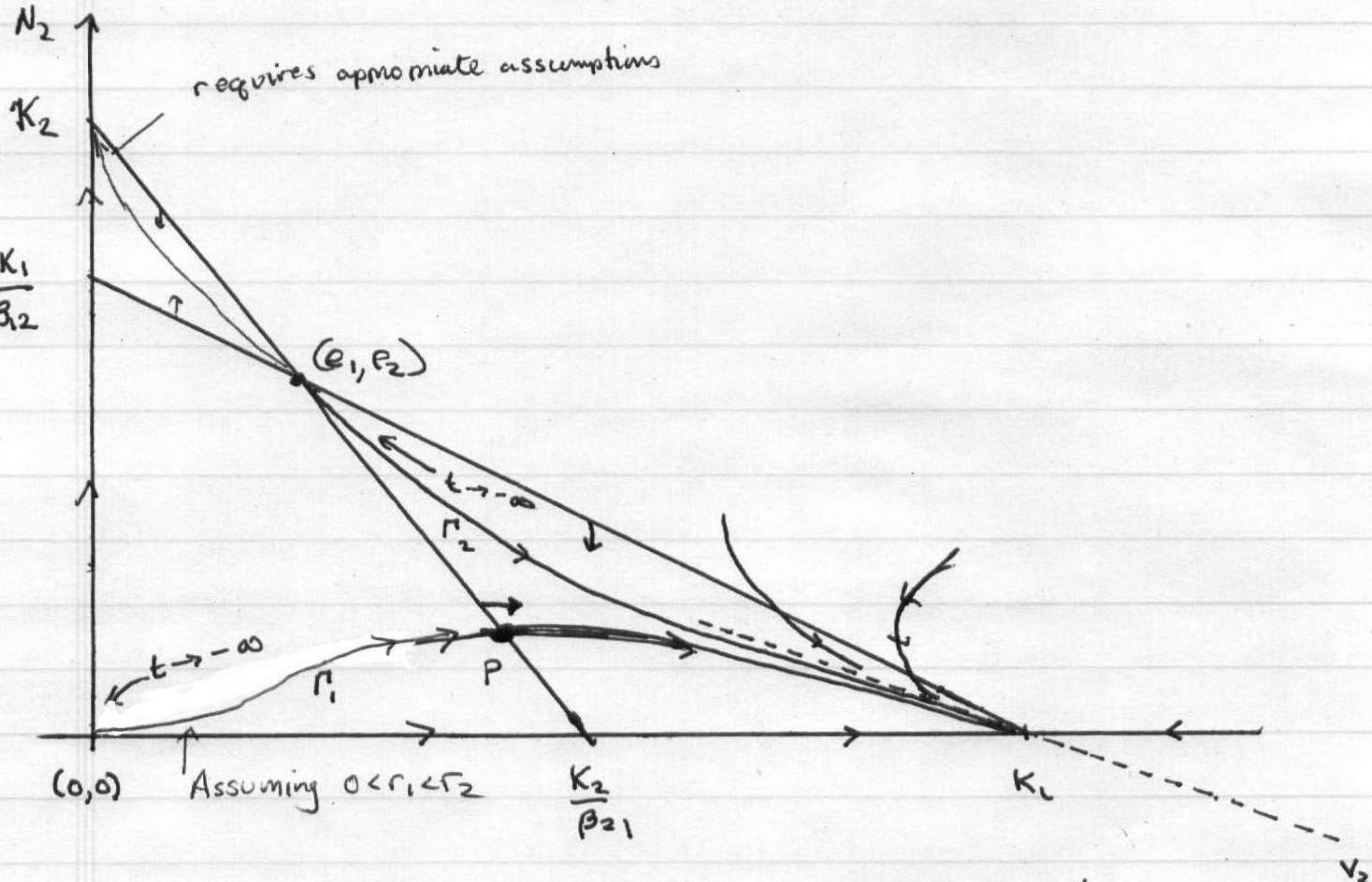
The eigenvector associated to $-r_1$ is $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

and the eigenvector associated to $-\delta$ is $v_2 = \begin{bmatrix} 1 \\ \frac{r_1 - \delta}{r_1}\left(-\frac{1}{\beta_{12}}\right) \end{bmatrix}$

The line through v_2 has slope $\frac{r_1-\delta}{r_1}\left(-\frac{1}{\beta_{12}}\right)$.

Let us consider the case $0 < \delta < r_1$. Then all solutions tend to $(K_1, 0)$ tangent to v_2 except the solution moving in the

N_1-axis. Also $0 < \frac{r_2-\delta}{r_1} < 1$ is more gently sloped than the N_1 cline intersecting $(K_1, 0)$. Thus the phase portrait will look as follows near $(K_1, 0)$



Explanation of Γ_1: Consider the solution curve passing through P. Suppose $(N_1(0), N_2(0)) = P$. For $t > 0$, it enters the sector under study and approaches $(K_1, 0)$ tangent to v_2. As $t \downarrow -\infty$ it can only move back ~~along~~ toward $(0,0)$.

Now

$$J(0,0) = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$$

Suppose $0 < r_1 < r_2$ ... r_1 and r_2 are eigenvectors with associated them are $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Solution curves approach $(0,0)$ (as $t \downarrow -\infty$) tangent to v_1

Γ_2 will be explained later.

A similar picture holds in the other triangular sector.

Next, we look at the Jacobian at (e_1, e_2). At this point $g_1(e_1,e_2) = 0 = g_2(e_1,e_2)$ and so

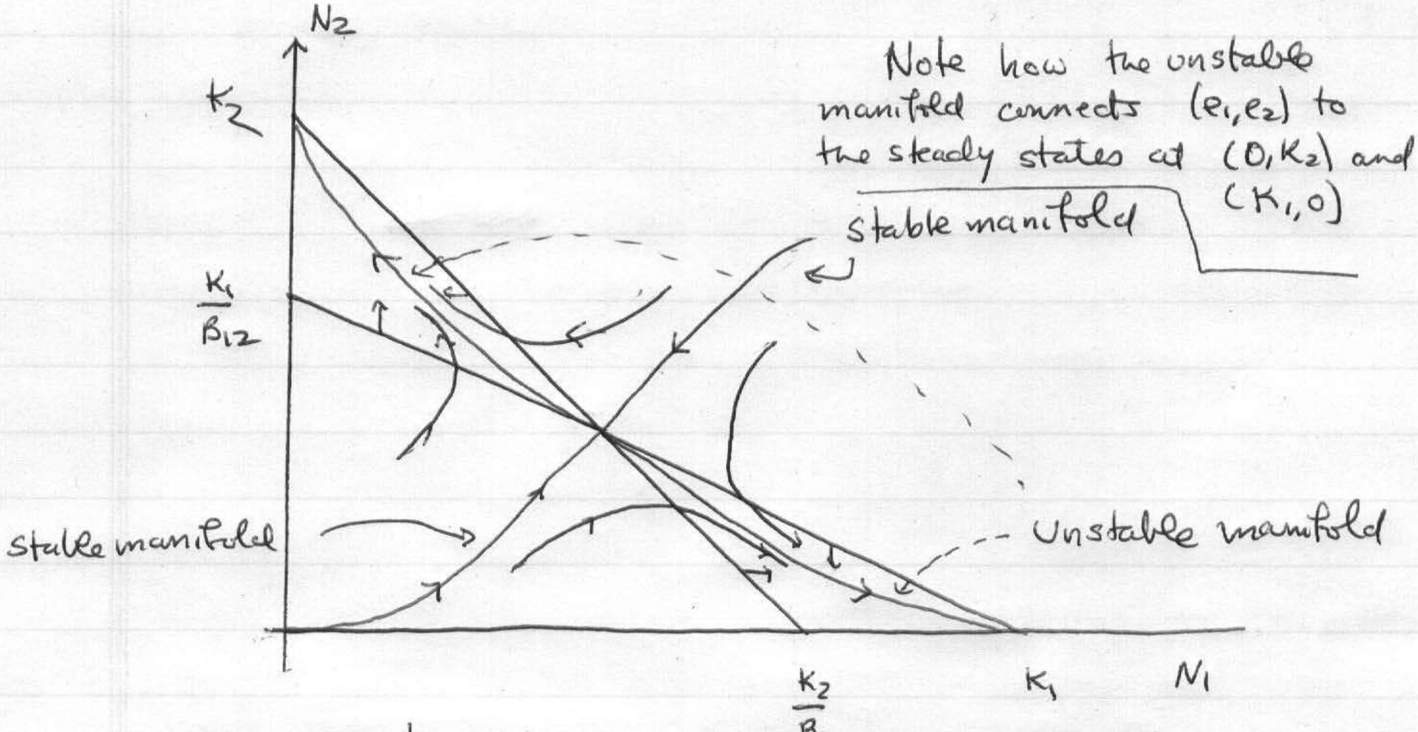
$$J(e_1,e_2) = \begin{bmatrix} -\frac{r_1 e_1}{K_1} & -\frac{r_1 \beta_{12}}{K_2} e_1 \\ -\frac{r_2 \beta_{21} e_2}{K_2} & -\frac{r_2}{K_2} e_2 \end{bmatrix}$$

Therefore $\quad \beta = \operatorname{tr} J(e_1,e_2) = -\left[\frac{r_1 e_1}{K_1} + \frac{r_2 e_2}{K_2}\right] < 0$

$$\gamma = \det J(e_1,e_2) = \frac{r_1 r_2}{K_1 K_2} e_1 e_2 \left(1 - \beta_{12}\beta_{21}\right) < 0$$

(remember $\beta_{12}\beta_{21} > 1$)

Thus (e_1,e_2) is a <u>saddle point.</u> It is clear that the stable and unstable manifolds have a form as shown



We have run the stable manifold in the lower section backwards in time and it connects to the unstable nodes at the origin