Math 336, Fall 2008, Linearization and stability of systems; outline

I. Stability of linear systems.

Let A be a $d \times d$-square matrix, and let x stand for a d-vector,

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}.$$

Consider the linear system of differential equations

$$\frac{d}{dt}x(t) = A \cdot x(t) \tag{1}$$

If $\det(A) \neq 0$, then the origin $\underline{0}$, which is the vector of all zero entries, is the unique equilibrium point for equation (1).

Theorem 1 *Suppose that* $\det(A) \neq 0$ *and suppose that*

$$\textit{the real part of every eigenvalue of } A \textit{ is strictly negative} \tag{2}$$

Then the origin $\underline{0}$ *is a* globally aymptotically stable *equilibrium point of system (1). This means that* $\lim_{t\to\infty} x(t) = \underline{0}$ *for every solution* $x(t)$ *of (1).*

The reason this theorem is true is that *every* solution of (1) may be written as a linear combination of vector-valued functions of the form $t^k e^{\lambda t} v$, where λ is an eigenvalue of A and v is a generalized eigenvector associated to λ. (A generalized eigenvector associated to λ is a nonzero vector v satisfying an equation of the form $(A - \lambda I)^j v = \underline{0}$, for some positive integer j.) If λ is real and $\lambda < 0$, then $\lim_{t\to\infty} t^k e^{\lambda t} = 0$ for any k. If $\lambda = a + bi$, where $a < 0$, then $t^k e^{\lambda t} = t^k e^{at}(\cos(bt) + i\sin(bt))$, and it is still true that $\lim_{t\to\infty} t^k e^{\lambda t} = 0$ for any k. Hence, when the eigenvalues of A all have negative real parts, all solutions must converge to $\underline{0}$ as time increases.

When A is the 2×2-matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

then the eigenvalues of A have negative real part if and only if

$$\mathrm{tr}(A) = a_{11} + a_{22} < 0 \quad \text{and} \quad \det(A) = a_{11}a_{22} - a_{12}a_{21} > 0. \tag{3}$$

Thus these two conditions imply stability for the 2×2 case.

II. Linearization of a 2-dimensional system at an equilibrium point.

Consider the nonlinear system of 2 differential equations

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} F(X,Y) \\ G(X,Y) \end{pmatrix}. \tag{4}$$

Let $(\bar{X}, \bar{Y})$ be an equilibrium point; that is, $F(\bar{x}, \bar{y}) = 0,\ G(\bar{X}, \bar{Y}) = 0$. The tangent plane linear approximation to

$$= \begin{pmatrix} F(X,Y) \\ G(X,Y) \end{pmatrix}$$

at $(\bar{X}, \bar{Y})$ is

$$= \begin{pmatrix} F_x(\bar{X},\bar{Y}) & F_y(\bar{X},\bar{Y}) \\ G_x(\bar{X},\bar{Y}) & G_y(\bar{X},\bar{Y}) \end{pmatrix} \begin{pmatrix} X-\bar{X} \\ Y-\bar{Y} \end{pmatrix}.$$

It follows then that if (X, Y) is a solution to (4),

$$\frac{d}{dt}\begin{pmatrix} X-\bar{X} \\ Y-\bar{Y} \end{pmatrix} \approx \begin{pmatrix} F_x(\bar{X},\bar{Y}) & F_y(\bar{X},\bar{Y}) \\ G_x(\bar{X},\bar{Y}) & G_y(\bar{X},\bar{Y}) \end{pmatrix} \begin{pmatrix} X-\bar{X} \\ Y-\bar{Y} \end{pmatrix},$$

when $X, Y)$ is close to $(\bar{X}, \bar{Y})$. The linearization of (4) at $\bar{X}, \bar{Y})$ is the linear differential equation obtained by replacing $\approx$ by $=$ in the previous equation:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} \approx \begin{pmatrix} F_x(\bar{X},\bar{Y}) & F_y(\bar{X},\bar{Y}) \\ G_x(\bar{X},\bar{Y}) & G_y(\bar{X},\bar{Y}) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{5}$$

Theorem 2 *(Sufficient condition for stability.) If the origin is asymptotically stable for the linear system (5), the equilibrium point $(\bar{X}, \bar{Y})$ is stable for the nonlinear system (4).*

The idea is that solution $(x(t), y(t))$ to (5) are good approximations to $X(t) - \bar{X}, Y(t) - \bar{Y})$, where $(X(t), Y(t))$ solves 4), when the origin is stable for (5), because stability implies that a solution to (5) that starts near $\underline{0}$ will stay near and even tend to $\underline{0}$, and in this region the right-hand side of (5) closely approximates that of (4).

III. Application to the chemostat model.

The chemostat model is

$$\frac{d}{dt}\begin{pmatrix} N \\ C \end{pmatrix} = \begin{pmatrix} F(N,C) \\ G(N,C) \end{pmatrix} = \begin{pmatrix} \alpha_1 \frac{C}{1+C} N - N \\ -\frac{C}{1+C} N - C + \alpha_2 \end{pmatrix}. \tag{6}$$

A calculation shows that

$$\begin{pmatrix} F_N(\bar{N},\bar{C}) & F_C(\bar{N},\bar{C}) \\ G_N(\bar{N},\bar{C}) & G_C(\bar{N},\bar{C}) \end{pmatrix} = \begin{pmatrix} \frac{\alpha_1 \bar{C}}{1+\bar{C}} - 1 & \frac{\alpha_1 \bar{N}}{(1+\bar{C})^2} \\ -\frac{\bar{C}}{1+\bar{C}} & -\frac{\bar{N}}{(1+\bar{C})^2} - 1 \end{pmatrix}$$

The chemostat has two equilibria. One occurs at

$$(\bar{N}_1, \bar{C}_1) = \left(\alpha_1\left[\alpha_2 - \frac{1}{\alpha_1 - 1}\right], \frac{1}{\alpha_1 - 1}\right).$$

The only interesting case physically is when both coordinates of this equilibrium point are positive, and this requires $\alpha_1 > 1$ and $\alpha_2 > 1/(\alpha_1 - 1)$, so these conditions will be assumed henceforth. A simple calculation shows that $\bar{C}_1/(1+\bar{C}_1) = 1/\alpha_1$. Using this, one finds

$$\begin{pmatrix} F_N(\bar{N}_1, \bar{C}_1) & F_C(\bar{N}_1, \bar{C}_1) \\ G_N(\bar{N}_1, \bar{C}_1) & G_C(\bar{N}_1, \bar{C}_1) \end{pmatrix} = \begin{pmatrix} 0 & \frac{\alpha_1 \bar{N}_1}{(1+\bar{C}_1)^2} \\ -\frac{1}{\alpha_1} & -\frac{\bar{N}_1}{(1+\bar{C}_1)^2} - 1 \end{pmatrix}$$

We could calculate the other two entries in this matrix in terms of α_1 and α_2, but we shall not need to do so for the stability analysis. Calling this matrix A,

$$\mathrm{tr}(A) = 0 - \frac{\bar{N}_1}{(1+\bar{C}_1)^2} - 1 < 0$$

since we are assuming α_1 and α_2 are such that $\bar{N}_1 > 0$. Also

$$\det(A) = \frac{\bar{N}_1}{(1+\bar{C}_1)^2} > 0.$$

According to (3), these inequalities imply that the real parts of the eigenvalues of A are negative and hence that the origin is a stable point of the linear system, $dx/dt = Ax$. It then follows from Theorem 2, that $(\bar{N}_1, \bar{C}_1)$ is a stable equilibrium of the chemostat model (6).