Math 336, Problem Set 8, Fall 2008

Read Edelstein-Keshet, Chapter 8.1-8.5.
Hand in the problem from Edelstein-Keshet below and also problem C.

Edelstein-Keshet, Chapter 8: 6(a)-(e);

A. Consider the Van der Pol equation:

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v - [\frac{u^3}{3} - u] \\ -u \end{pmatrix}$$

(a) Use Bendixon's criterion to show that there no periodic solution to this equation entirely contained in the half plane $\{u > 1\}$.

(b) Similarly show that there is no periodic solution entirely contained in the strip $\{-1 < u < 1\}$.

B. For which of the following equations is the unit square $[0, 1] \times [0, 1]$ forward invariant, meaning any solution starting in the unit square must remain inside it for all future times. You must check the direction field on each of the four sides. It may be convenient to use the criterion that the direction field points into the region if the inner product of the direction field with an vector normal to the boundary and pointing out is less than or equal to zero.

(a) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y - x \\ x - y \end{pmatrix}.$

(b) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y(x-1) \\ -x \end{pmatrix}.$

(c) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\pi(x + y/8)) \\ x^2 - y^2 \end{pmatrix}.$

(d) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy(x-1) \\ -xy - y \end{pmatrix}.$

(e) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy(x-1) \\ -xy - 1 \end{pmatrix}.$

C. Consider the Van der Pol equation

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v - [\frac{u^3}{3} - u] \\ -4u \end{pmatrix}$$

Use the Poincaré-Bendixon theorem to find a region about the origin that contains a periodic orbit. Follow the strategy we used in class for the Van der Pol equation in problem A. Look for a region whose boundary in the right half plane consists of a horizontal line between the positive v-axis and the u-cline a vertical line from the u-cline to a line of the form $v = -A + \beta u$, where $\beta > 0$. We know that the direction field points into the region along the top horizontal line and the vertical line on the side. The problem is to find a good $-A$ and β so that the direction field will also point in on the lower slanted line.

(a) First choose a good β. Show that an outward normal to the line with slope β is the vector $N = \langle 1, -1/\beta \rangle$. Compute $N \cdot \langle v - [\frac{u^3}{3} - u], -4u \rangle$ along the line $v = -A + \beta u$ and show that you get an expression of the form $-A + h(\beta)u + u - (u^3/3)$, where $h(\beta)$ is an function of β. Give the explicit form of h. To keep the contribution of $h(\beta)$ as small as possible, we want to minimize it with respect to β. Show that it is minimized for $\beta > 0$ at $\beta = 2$.

(b) Now setting $\beta = 2$, find an A that ensures $N \cdot \langle v - [\frac{u^3}{3} - u], -4u \rangle \leq 0$ along the line $v = -A + 2u$ for $u \geq 0$.

(c) Fit together a region that must contain a periodic orbit using the horizontal, vertical and sloped lines we have discussed.