B. Problem on the Fitzhugh-Nagumo equation

Consider the equation

$$\frac{dx}{dt} = y - \left[\tfrac{1}{2}x^3 - x\right] + i$$

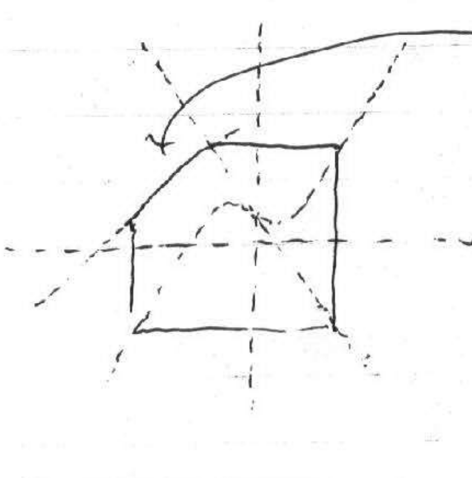
$$\frac{dy}{dt} = -\left[x + \tfrac{1}{2}y - \tfrac{3}{4}\right]$$

(A calculator will be useful in this problem)

a) Let $i=0$. Find the steady state. Show directly that it is stable. Plot the nullclines and show that this plot corresponds to case I of the class notes on the Fitzhugh-Nagumo equation (available online).

b). Let $i=-\frac{1}{3}$. Find the steady state and determine whether it is stable or not. Plot the null clines. Use JODE or MAPLE to plot the solution that passes through the steady state of part (a)

c) Let $i=-3/2$. Find the steady state and show that it is unstable. Prove the existence of a limit cycle by using the Poincaré-Bendixon theorem. Hint: you need to show the existence of a bounded, forward invariant (trapping) region containing the unstable steady state. By using symmetry conditions you can show the existence of a triangular rectangular region. If you take this route, be sure to justify your construction and show that the region is forward invariant. Alternatively you can look for a region of the form



This line has equation $y = x + A$, $A>0$. Find a suitable and then specify how to construct the other boundaries.

d) Plot solutions to the case $i = -3/2$ with the help of JOde or MAPLE. and verify the existence of a stable limit cycle.

e). Show that for the case $i = -3/2$, any solution that starts on the positive y-axis does not cross the line segment defined by $y = 2x - 5$, $x \geq 0$.

f) (Graphing calculator) Let $i = -3/2$. Show that a solution that starts on the positive y-axis will not cross the curve defined by $y = \frac{1}{2}(x^2 - 4)$, $x \geq 4$.