336 NOTES, 2008: SCALAR ODE's, REVIEW

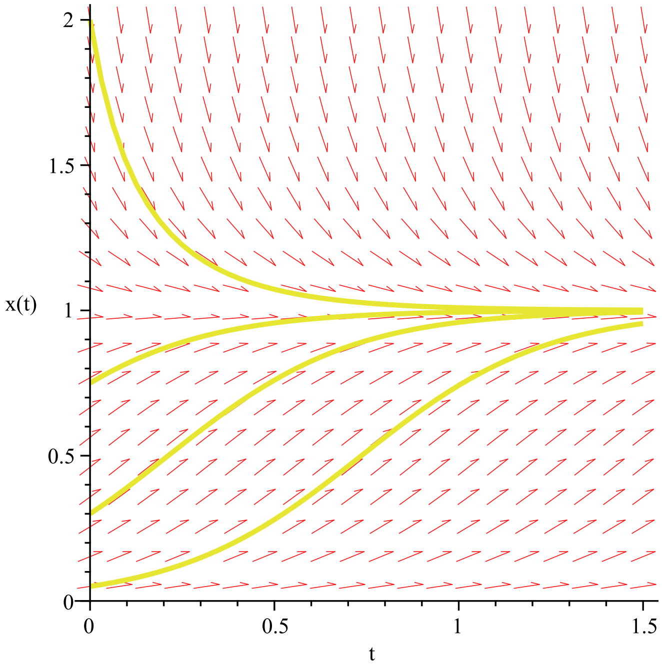
This is a review of some techniques for understanding solutions to differential equations of the form

$$\frac{dx}{dt} = f(t, x), \tag{1}$$

where x is a scalar variable. We are particularly interested in the autonomous case in which f does not depend explicitly on t:

$$\frac{dx}{dt} = f(x). \tag{2}$$

Solutions to these equations can be visualized qualitatively and graphically using *direction fields*. To obtain a direction field for equation (1), we choose a grid of points in the (t, x)-plane, and for each point (t_i, x_i) in this grid we draw a little arrow through (t_i, x_i) whose slope is $f(t_i, x_i)$ and which points in the direction of positive t. What is the point of this? Imagine graphing a solution $x(t)$ of (1) in the (t, x)-plane that passes through (t_i, x_i). The derivative dx/dt at (t_i, x_i) is the slope of the tangent line to the graph at (t_i, x_i) and equation (1) says this slope must equal $f(t_i, x_i)$. Hence the solution curve must be tangent to the little arrow of the direction field at (t_i, x_i). Thus, if the grid of points of the direction field is fine enough, we can get a good rough idea of the shape of solutions by sketching in curves that intersect tangent to all arrows, or all directions, of the direction field.



The figure on the previous page shows a direction field for the logistic equation

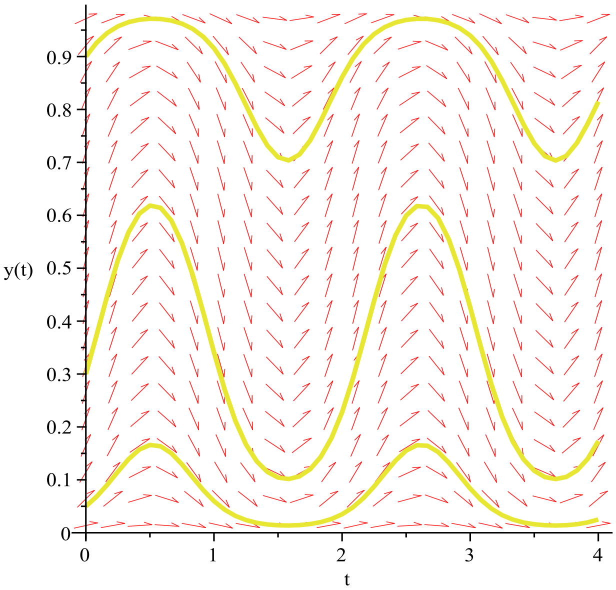
$$\frac{dx}{dt} = 4x(1-x), \tag{3}$$

Sample solution curves are also shown, corresponding to solutions with initial coniditions, $x(0) = 0.05$¡ $x(0) = 0.3$, $x(0) = 0.75$, and $x(0) = 2$.

The following figure shows a direction field with selected solution curves for a non-autonomous variation of equation (3):

$$\frac{dx}{dt} = 4x(1-x)\cos(3t), \tag{4}$$

These figures were obtained by using the *DEplot* command in Maple.



It is not hard to determine the qualitative behavior of solutions to autonomous equations, even without drawing a direction field, because the directions of the direction field do not vary with t. We will state the principles for this analysis, under the important assumptions that f in equation (2) is continuous and the solutions to (2) with initial condition $x(t_0) = x_0$ is unique for any values of t_0 and x_0. (There is an important theorem whose statement you should know: If f is continuously differentiable, solutions to equation (2) are unique.)

In the qualitative analysis of solutions, *equilibrium points* play a crucial role. An equilibrium point for (2), also called a *steady state point*, or a *singular point*, is a point r such that $f(r) = 0$. If r is such a point, then the constant function $x(t) \equiv r$

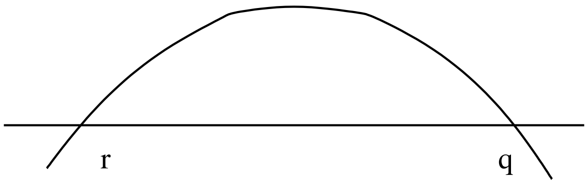
solves equation (2). It represents an equilibrium state of whatever physical variable x models. An equilibrium point r is called *asymptotically stable* if solution curves in the vicinity of r all converge to r as $t \to \infty$; more precisely, r is asymptotically stable if there is a neigborhood $a < r < b$ such that for any x_0 in (a, b), if x is a solution with $x(t_0) = x_0$ for some t_0, then $\lim_{t\to\infty} x(t) = r$. In the first figure, it is clear that $r = 1$ is an asymptotically stable equilibrium point. Asymptotically stable equilibria are also sometimes called *attractors*.

The principles are:

(i) Solutions curves do not intersection. In particular, no non-constant solution can pass through an equilibrium point.

(ii) If x is a solution and $z = \lim_{t\to\infty} x(t)$ exists and is finite, then z is an equilibrium point. Similarly, if $w = \lim_{t\to-\infty} x(t)$ exists and is finite, then w is an equilibrium point.

How are these applied? Suppose that r and q are two consecutive roots of f and $r < q$; thus r and q are equilibria and there are no equilibria between them. Since f is continuous, it is either always positive or always negative on (r, q). Let us suppose it is positive, as in the following figure.



Now suppose that $x(t)$ is a solution of the differential equation passing through the point (t_0, x_0), where $r < x_0 < q$. Then by principle (i) the graph of $x(t)$ cannot cross either r or q, so the solution will remain between these two equilibria for *all times*, positive and negative. Moreover, in this region $f(x) > 0$ and hence $dx/dt = f(x(t)) > 0$ for all t. This means that as t increases, $x(t)$ is increasing. Now since $x(t)$ increases as a function of t and never can get above q, $z = \lim_{t\to\infty} x(t)$ must exist and $r < z < q$. However, by principle (ii) z must be an equilibrium point, and since there are no equilibrium points strictly between r and q, it follows that $\lim_{t\to\infty} x(t) = q$. By a similar argument. $\lim_{t\to-\infty} x(t) = r$. This is precisely the situation we see illustrated in the first example. Here $f(x) = 4x(1 - x) > 0$ for $0 < x < 1$, $r = 0$ and $q = 1$, and one sees from the direction field and the solution plots that all solutions starting out between 0 and 1 will converge to the upper boundary $q = 1$ as $t \to \infty$.

If, instead, $f(x) < 0$ between consecutive equilibria, the same type analysis shows that as $t \to \infty$, $x(t)$ decreases to the lower equilibrium point as $t \to \infty$.

What happens if, say r is an equilibrium point and $f(x) < 0$ for all $x > r$? This is easy, $x(t)$ must decrease as long as it stays in the region greater than r. But it cannot cross below r and hence $\lim_{t\to\infty} x(t) = r$. This is precisely the situation of the first example for solutions starting out above level 1. If, on the other hand, $f(x) > 0$ for all $x > r$, solutions in the region above r must always increase. Then there are two possibilities. First, the solution could be defined for all positive times and increase to ∞ as $t \to \infty$. Second, the solution could have an explosion; this means there is a $t_f < \infty$ such $\lim_{t\uparrow t_f} x(t) = \infty$. Determining which of these two case happens requires further analysis, although sometimes one can guess by looking at the direction field. A sufficient condition that the solution does *not* explode is that $|f(x)| < K(1 + |x|)$ for some positive constant K.

Using the analysis we have just done, it is easy to identify the asymptotically stable equilibria. An equilibrium point r is asymptotically stable if $f(x) > 0$ immediately to the left of r and $f(x) > 0$ immediately to the right of r. You should justify to yourself why this is so, in general. Also check that this applies to the first example to show that 1 is asymptotically stable, as we have already remarked.

It is also not hard to determine where the solution curves are concave up or down. Remember that $x(t)$ will be concave up where its second derivative is positive and concave down where its second derivative is negative. But if x solves equation (2) then, using the chain rule,

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}f(x(t)) = f'(x(t))\frac{dx}{dt} = f'(x(t))f(x(t)). \tag{5}$$

Thus, $x(t)$ is concave up when it is in a region where $f'(x)f(x) > 0$ and concave down when it is in a region where $f'(x)f(x) < 0$.

Finally, let us briefly, if informally, justify our two principles. The first is true simply because of the assumption of the the uniqueness of solutions. If two, different solution curves intersected at a point (t_0, x_0) then there would be two different solutions with this initial condition. As for the second principle, imagine that $z = \lim_{t\to\infty} x(t)$ exists and is finite. Since $x(t)$ is settling down to a constant value and since the derivative of a constant function is zero, it ought to be true that $0 = \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} f(x(t))$; it can be shown that this is indeed the case, but we do not give the argument. Then, since f is assumed continuous, $f(z) = \lim_{t\to\infty} f(x(t)) = f\left(\lim_{t\to\infty} x(t)\right) = 0$, and so z is an equilibrium point.