Math 336, Lecture Notes, Nov. 6

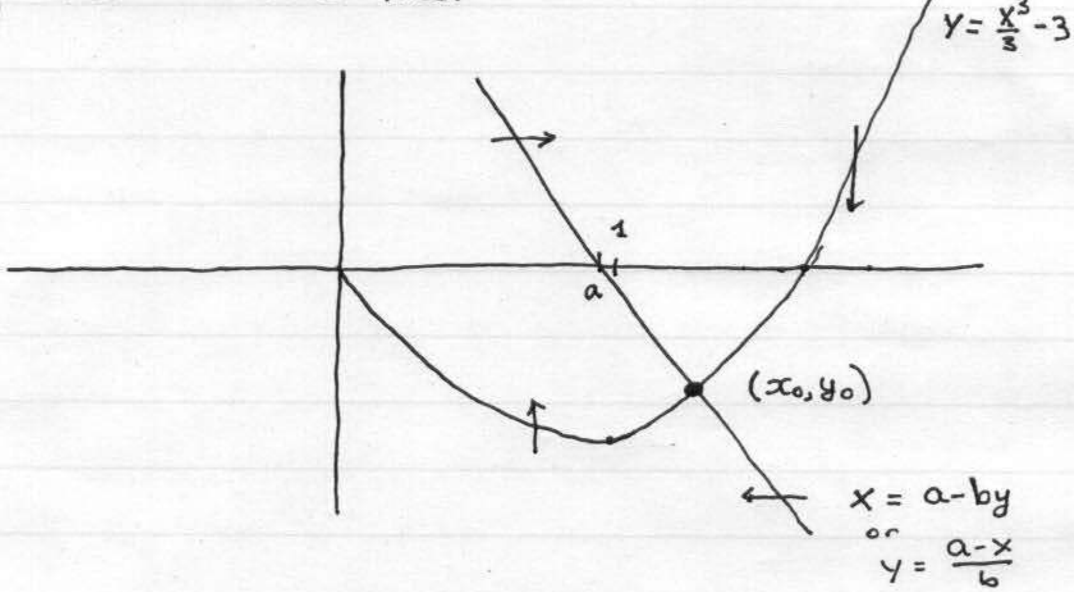
Fitzhugh - Nagumo Equation

$$\frac{dx}{dt} = c\left[y - \left[\frac{x^3}{3} - x\right] + i(t)\right]$$

$$\frac{dy}{dt} = -\frac{1}{c}\left[x - (a - by)\right]$$

Here we assume $0 < b < 1$ and $b < c^2$. We are interested mostly in the case when c is relatively large ($c^2 > 1 > b$ automatically) so that the y variable is relatively a slow variable.

Case I. $i \equiv 0$, $a > 0$ is chosen so that the graph of the x- and y- null clines looks like:



In this case,

(x_0, y_0) is a stable spiral (focus) steady state

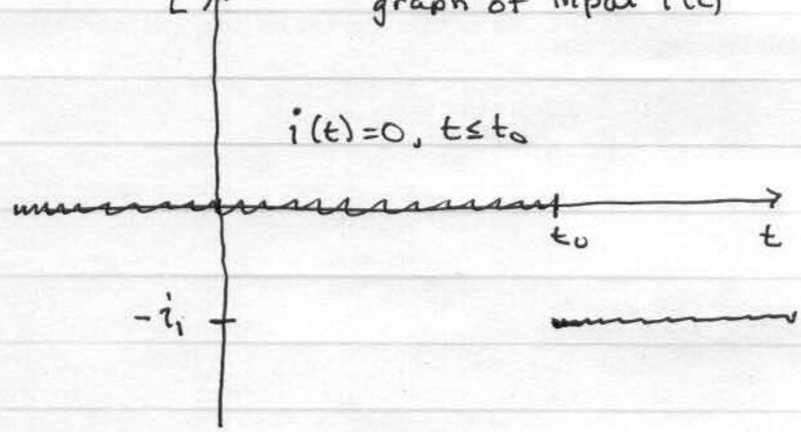
because

$$J(x_0, y_0) = \begin{bmatrix} -c \left.\frac{d}{dx}\left(\frac{x^3}{3} - x\right)\right|_{x_0} & c \\ -1/c & -\frac{b}{c} \end{bmatrix}$$

and from the graph $\left.\frac{d}{dx}\left[\frac{x^3}{3} - x\right]\right|_{x = x_0} = x_0^2 - 1 > 0$, so that $\operatorname{tr} J(x_0, y_0) = -c(x_0^2 - 1) - \frac{b}{c} < 0$ and $\det J(x_0, y_0) = b(x_0^2 - 1) + 1 > 0$.

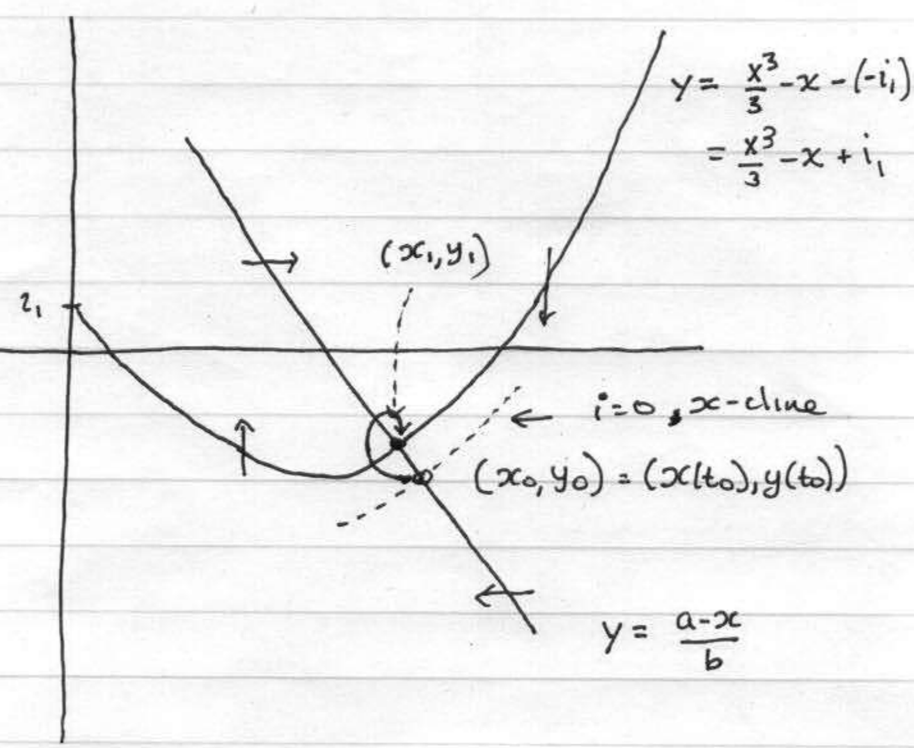
Case II : $i \equiv -i_1$ where $i_1 > 0$ is small enough that the new steady state (x_1, y_1) still satisfies $x_1 > 1$. By the same calculation as before, (x_1, y_1) is still a stable spiral (focus).

Question. Suppose $i \equiv 0$ and the solution has had time to approach the steady state (x_0, y_0) closely. To a good approximation we can assume that we are at the steady state at some time t_0: $(x(t_0), y(t_0)) \simeq (x_0, y_0)$. Now the input i is stepped down to $-i_1$. What happens?



The picture at time $t = t_0$ for the new system (with $i = -i_1$) is as follows:

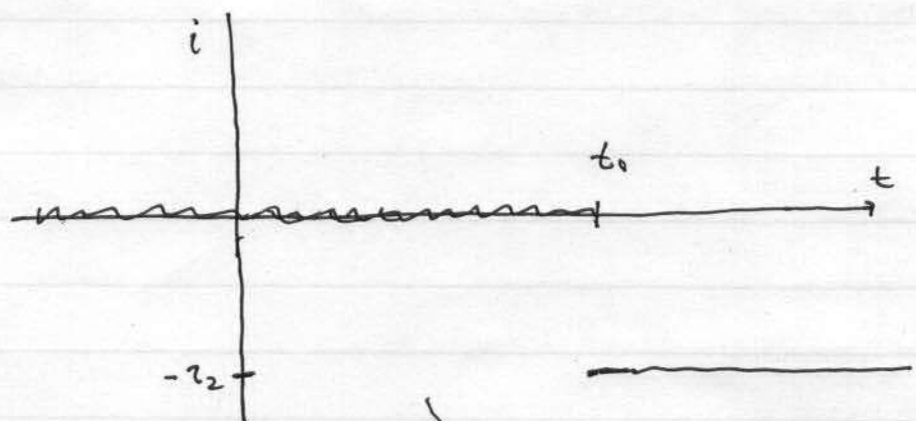
The steady state has shifted to (x_1, y_1) but is still stable. The solution begins to move from (x_0, y_0) in the northwesterly direction because that is the trend of the direction field when $i = -i_1$. Since the new x-cline is close by, it reaches it soon and converges to (x_1, y_1) making only a small excursion, as shown.

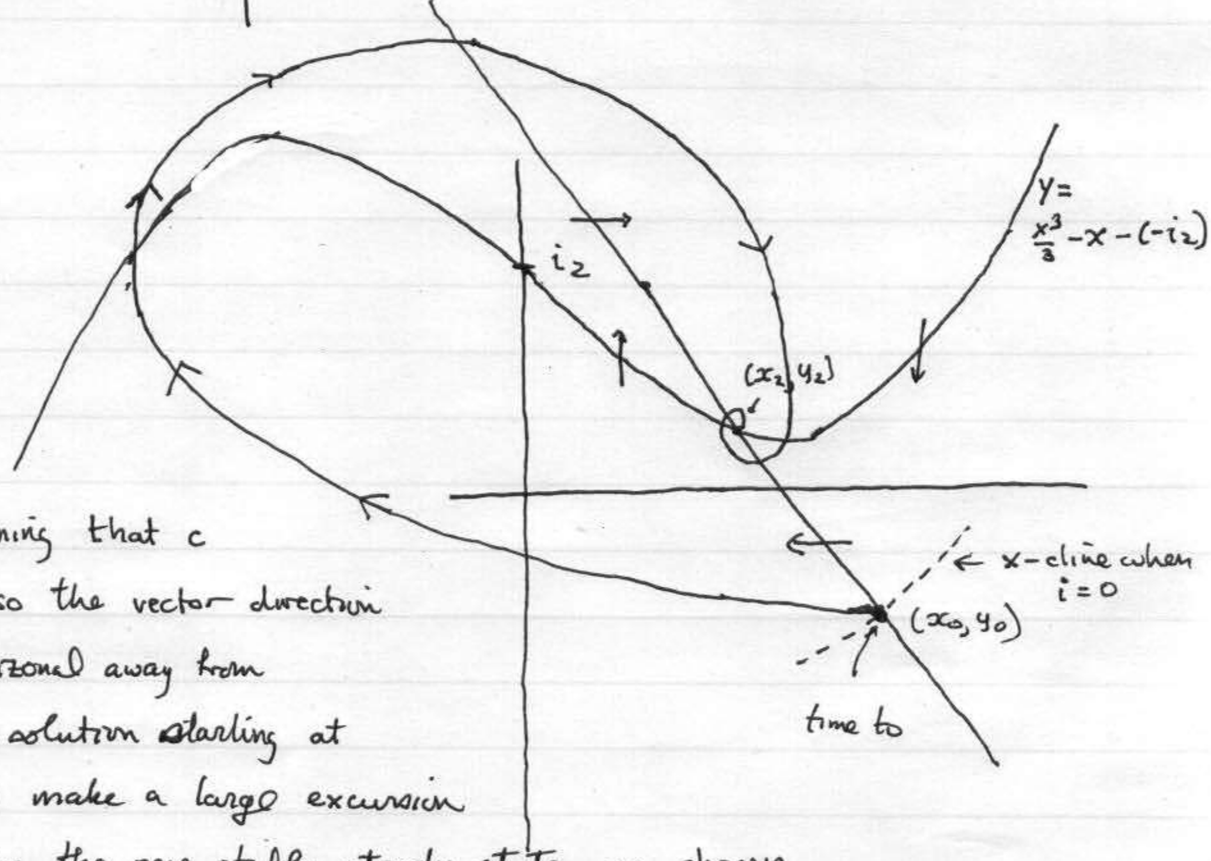


<u>Case III</u>. $i = -i_2$, where i_2 is such that now for the new steady state (x_2, y_2) $\quad x_2 < 1$ <u>but</u> $x_2^2 - 1 > -\frac{b}{c^2}$

Then $\operatorname{tr} J(x_2, y_2) = -c(x_0^2 - 1) - \frac{b}{c} < 0$ still; because $0 < b < 1$ and $x_0^2 - 1 > -1$, $\det J(x_2, y_2) = b(x_0^2 - 1) + 1 > -b + 1 > 0$. Hence (x_2, y_2) is still a stable steady state.

Now ask the same question as before but with $i(t)$ stepping to $-i_2$ at t_0

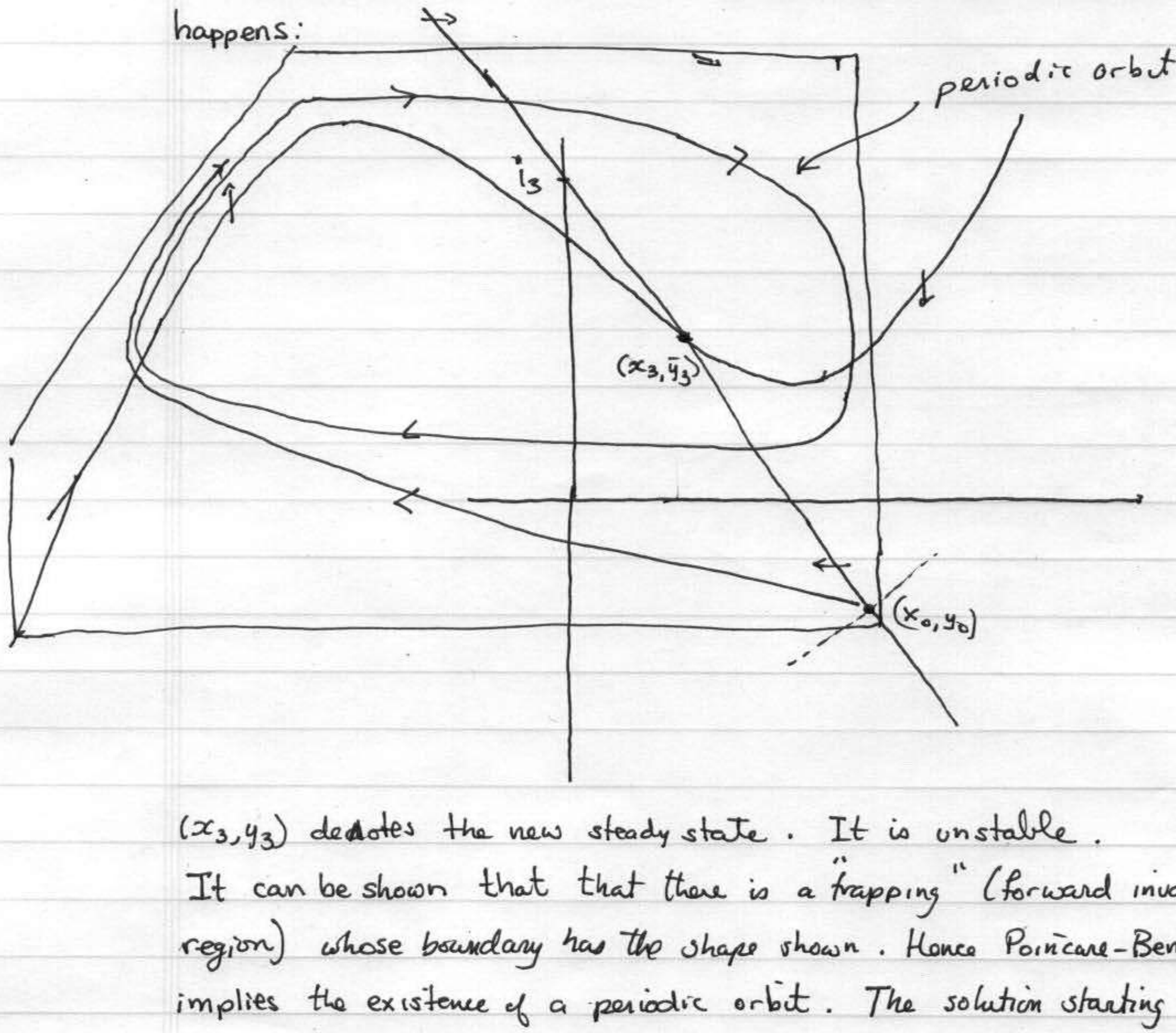




If we are assuming that c is large enough so the vector direction field is fairly horizontal away from the x-cline, the solution starting at (x_0, y_0) will now make a large excursion before approaching the new stable steady state, as shown.

A system exhibiting this behavior is called <u>excitable</u>. Such an excitable response provides a caricature of an action potential spike generated by an input voltage in a neural impulse.

Case IV. Since $\frac{b}{c^2}$, $-\frac{b}{c^2} > -1 = (x_0^2-1)\Big|_{x=0}$. Therefore if $-i$ is large enough, then at the steady state $(\bar{x},\bar{y})$ associated to i $\operatorname{tr} J(\bar{x},\bar{y}) = -b(\bar{x}_0^2-1) - \frac{b}{c^2} > 0$ and this steady state will no longer be stable. Let $i = -i_3$ be large enough so that this happens:



(x_3,y_3) denotes the new steady state. It is unstable. It can be shown that that there is a "trapping" (forward invariant region) whose boundary has the shape shown. Hence Poincare-Bendixson implies the existence of a periodic orbit. The solution starting at (x_0,y_0) under the influence of $i(t) = -i_3$ for $t \geq t_0$ will tend to this orbit. This oscillatory behavior is a caricature of the repetitive firing of neural impulses under a large enough constant stimulus.