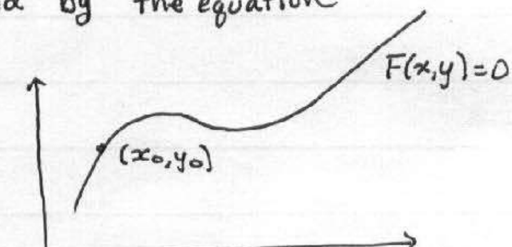
336 Class Notes

Implicit differentiation

Consider a curve in the plane defined by the equation

$$F(x,y) = 0 \qquad (1)$$



Suppose we want to find the slope $\left.\frac{dy}{dx}\right|_{(x_0,y_0)}$ of the tangent to the curve at (x_0,y_0), assuming it exists. To do this, differentiate (1) w.r.t x:

$$0 = \frac{d}{dx}F(x,y) = F_x(x,y)\frac{dx}{dx} + F_y(x,y)\frac{dy}{dx}$$
$$= F_x(x,y) + F_y(x,y)\frac{dy}{dx}$$

where $F_x(x,y) = \frac{\partial F}{\partial x}(x,y)$ and $F_y(x,y) = \frac{\partial F}{\partial y}(x,y)$. Solving

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)} \quad \text{along the curve.}$$

So
$$\left.\frac{dy}{dx}\right|_{(x_0,y_0)} = -\frac{F_x(x_0,y_0)}{F_y(x_0,y_0)}$$

Jacobians at steady states and slopes of null clines

Consider the system $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$

with null clines $0 = F(x,y)$ and $0 = G(x,y)$ that intersect at the steady state (x_0,y_0). Let

$$S_F(x_0,y_0) = \frac{-F_x(x_0,y_0)}{F_y(x_0,y_0)} = \text{slope of } x\text{-cline at } (x_0,y_0)$$

$$S_G(x_0,y_0) = -\frac{G_x(x_0,y_0)}{G_y(x_0,y_0)} = \text{slope of } y\text{-cline at } (x_0,y_0)$$

Therefore, the Jacobian at (x_0, y_0) is

$$J(x_0,y_0) = \begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ & \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix} = \begin{bmatrix} -S_F(x_0,y_0)\,F_y(x_0,y_0) & F_y(x_0,y_0) \\ & \\ -S_G(x_0,y_0)\,G_y(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix}$$

(assuming $F_y(x_0,y_0) \neq 0$, $G_y(x_0,y_0) \neq 0$)

and
$$\operatorname{tr} J(x_0,y_0) = -S_F(x_0,y_0)F_y(x_0,y_0) + G_y(x_0,y_0)$$
$$\det J(x_0,y_0) = F_y(x_0,y_0)G_y(x_0,y_0)\big(S_G(x_0,y_0) - S_F(x_0,y_0)\big)$$

The equations can be used to relate stability properties of the steady state to qualitative properties of the null clines and how they intersect.

<u>Example 1</u>. Suppose $F_y(x_0,y_0) > 0$ and $G_y(x_0,y_0) > 0$. This corresponds to the situation where $F > 0$ "above" the x-cline and $G > 0$ "above" the y-cline, as in each of the situations below

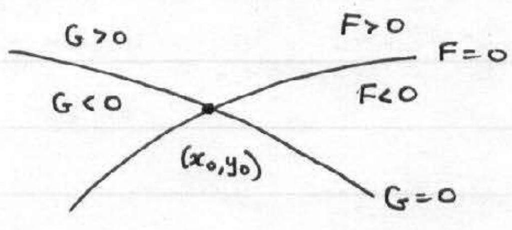


Figure 1 (a)

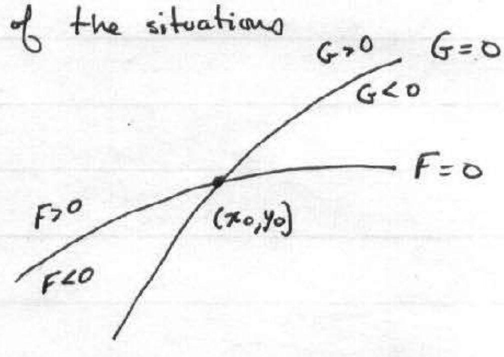


Figure (b)

In case (a), it is apparent from the picture that $S_F(x_0,y_0) > S_G(x_0,y_0)$. Hence it follows that $\det J(x_0,y_0)$
$= F_y G_y (S_G - S_F)\big|_{(x_0,y_0)} < 0$ and that (x_0,y_0) is a saddle point.

In case (b), $S_G(x_0,y_0) > S_F(x_0,y_0)$ and hence $\det J(x_0,y_0) > 0$. The steady state will be stable if

$$\operatorname{tr} J(x_0,y_0) = -S_F(x_0,y_0)\,F_y(x_0,y_0) + G_y(x_0,y_0) < 0$$

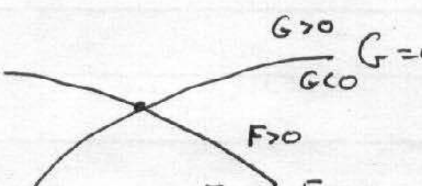
and will be an unstable node or spiral if

$$-S_F(x_0,y_0)\,F_y(x_0,y_0) + G_y(x_0,y_0) > 0$$

That is

$$(x_0,y_0) \text{ is } \begin{cases} \text{stable} & \text{if } G_y(x_0,y_0) < S_F(x_0,y_0)\,F_y(x_0,y_0) \\ \text{unstable node or spiral} & \text{if } G_y(x_0,y_0) > S_F(x_0,y_0)\,F_y(x_0,y_0) \end{cases}$$

If $S_F(x_0,y_0) < 0$. (which, since we are assuming $F_y(x_0,y_0) > 0$, is equivalent to $F_x(x_0,y_0) < 0$) we see right away that the second case holds and (x_0,y_0) is an unstable node or spiral.



If $S_F(x_0,y_0) > 0$, then whether (x_0,y_0) is stable or not depends on the magnitude of $G_y(x_0,y_0)$ relative to $S_F(x_0,y_0)\,F_y(x_0,y_0)$ and cannot be deduced from the plot of the null clines alone.

Example 2. The second example relates to Kolmogorov's criteria for the existence of a limit cycle in predator-prey models of the form

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\,f(x,y) \\ y\,g(x,y) \end{bmatrix} = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$$

In this model, x represents the level (or density) of the prey.

population, while y represents that of the predatore population.

The null clines in the interior of the quadrant $\{x \geq 0, y \geq 0\}$ are defined by $f(x,y)=0$ and $g(x,y)=0$, so at a steady state (x_0, y_0) with $x_0 > 0$, $y_0 > 0$.

$$S_F(x_0,y_0) = -\frac{F_x(x_0,y_0)}{F_y(x_0,y)} = \frac{-\overset{\nearrow 0}{f(x_0,y_0)} + x_0 f_x(x_0,y_0)}{x_0 f_y(x_0,y_0)}$$

$$= -\frac{f_x(x_0,y_0)}{f_y(x_0,y_0)} \qquad (2)$$

and similarly $S_G(x_0,y_0) = s_g(x_0,y_0) = {-g_x(x_0,y_0)}/{g_y(x_0,y_0)}$ (3). In fact (2) is true along the entire x-cline; (3) is true along the entire y-cline

For such a model it is reasonable to assume that

$$\frac{\partial}{\partial x} G(x,y) = g_x(x,y) > 0, \text{ or equivalently}$$

$$g_x(x,y) > 0 \quad \text{for} \quad x > 0 \ y > 0 \qquad (4)$$

This says that the growth rate of predators increases if the prey population increases.

Also the growth rate of prey should decrease whenever the predator population increases: that is

$$\frac{\partial}{\partial y} x f(x,y) = x f_y(x,y) < 0 \quad \text{or equivalently} \quad \underset{\text{for } x>0 \ y>0}{f_y(x,y) < 0}. \qquad (5)$$

We can expect also that the curve $g(x,y)=0$ is positively sloped. For each prey level x, the corresponding $y^{(x)}$ such that $g(x, y^{(x)})=0$ is the level of predator such that above for $y > y^{(x)}$ $g(x,y) < 0$ -- that is the predator has exceeded the carrying capacity for the given level x of prey. As x increases, so too should the carrying capacity $y^{(x)}$. This says precisely that along the curve $g(x,y)=0$ y increases as x increases. This means that along the y-cline

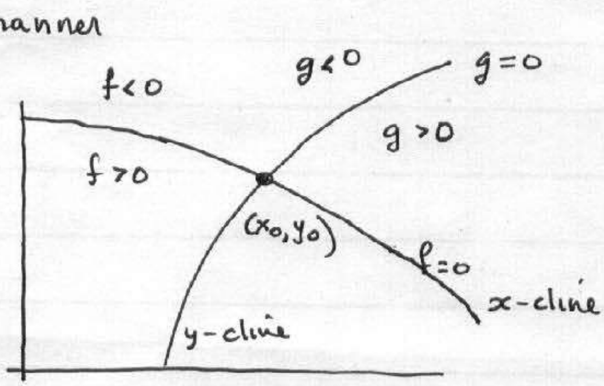
$$S_G(x,y) = \frac{-g_x(x,y)}{g_y(x,y)} > 0$$

which, since $g_x(x,y) > 0$ implies $g_y(x,y) < 0$ along the y-cline. In fact, it is reasonable to assume that

$$g_y(x,y) < 0 \quad \text{for} \quad x>0 \;\; y>0 \; ; \qquad (6)$$

namely, the per capita birth rate of the predator decreases as y increases.

Assume then that (4), (5), (6) are all valid and that the x-cline and y-cline intersect in the following manner



Note $f<0$ above $\{f=0\}$ and $f>0$ below $\{f=0\}$, according to the assumption (5) that $f_y < 0$.

Likewise because of (6) $g<0$ above the y-cline and $g<0$ below the y-cline.

In this picture $S_F(x_0,y_0) < 0 < S_G(x_0,y_0)$,

and $F_y(x_0,y_0) = x_0 f_y(x_0,y_0) < 0$ (remember that $F(x,y) = x f(x,y)$)

$G_y(x_0,y_0) = y_0 g_y(x_0,y_0) < 0$ (since $G_y(x_0,y_0) = \cancelto{0}{g(x_0,y_0)} + y_0 g_y(x_0,y_0)$)

Then

$$\operatorname{tr} J(x_0,y_0) = -S_F(x_0,y_0)\,y_0 f_y(x_0,y_0) + y_0 g_y(x_0,y_0)$$

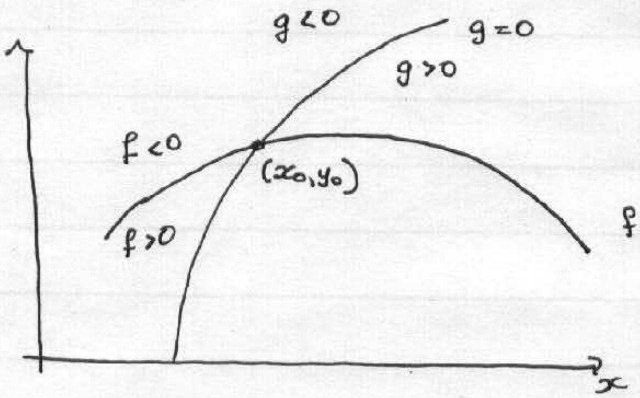
$$< 0 \qquad \text{since both } -S_F(x_0,y_0)\,y_0 f_y(x_0,y_0) \text{ and } y_0 g_y(x_0,y_0) \text{ are negative}$$

$$\det J(x_0,y_0) = x_0 y_0 f_y(x_0,y_0)\,g_y(x_0,y_0)\left[S_G(x_0,y_0) - S_F(x_0,y_0) \right]$$

$$> 0 \qquad \text{since } S_G(x_0,y_0) > S_F(x_0,y_0) \text{ and } f_y(x_0,y_0)\,g_y(x_0,y_0) > 0$$

In this case, the steady state is stable.

Consider instead the picture.



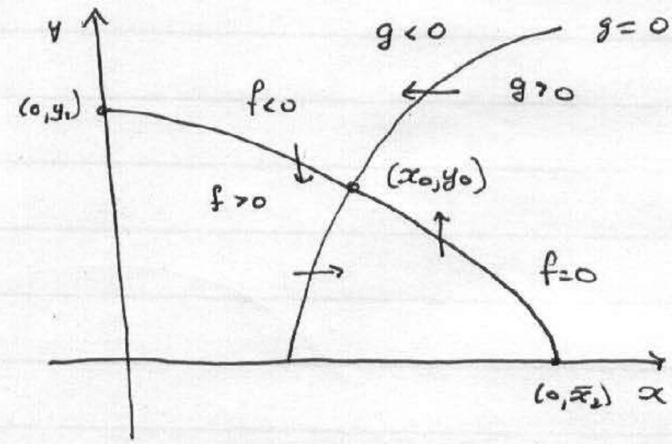
We still have $\det J(x_0,y_0) > 0$ since $S_G(x_0,y_0) > S_F(x_0,y_0)$

However, now $-S_F(x_0,y_0)\,y_0 f_y(x_0,y_0) > 0$ because $S_F(x_0,y_0) > 0$

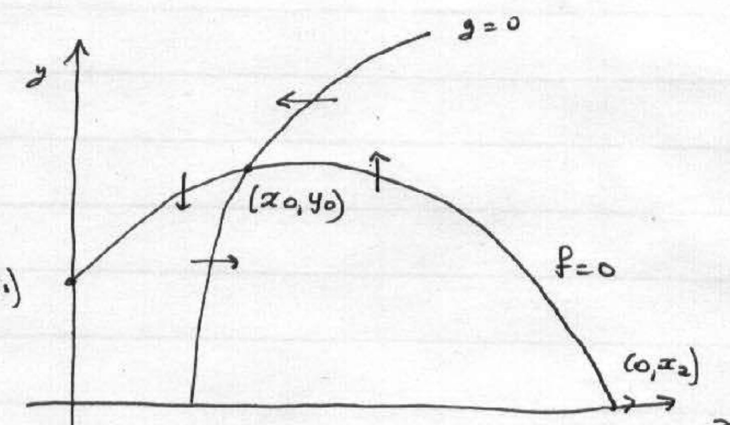
By itself this is not enough to imply (x_0,y_0) is an unstable node or spiral, but it is certainly necessary.

It does not make sense in the predator-prey model to assume that the null clines intersect so that $S_F(x_0,y_0) > S_G(x_0,y_0)$. There should be a level X so that $S_F(x,y) < 0$ for $x > X$ along the x-cline and a point $(0,x_2)$ on the x-axis with $f(0,x_2) = 0$. $(x_2 > X)$. The idea is that there is a level of prey population beyond which prey growth is negative.

even for small or zero levels of prey. In others for small y or $y=0$, there is a carrying capacity $x^{(s)}$ for the prey population. As the predator population increases this "carrying capacity" should decrease, at least at first (and for high enough predator levels $f(x,y)<0$ for all x so the x-cline is bounded above). Finally for small levels of x we expect $f(x,y)>0$ ~~to be~~ to hold, but for large y we expect $f(x,y)<0$ so that the prey population cannot grow. Fitting these ideas ~~to~~together we can posit that the x-cline has the form (in the region $X>0$, $y>0$)



or

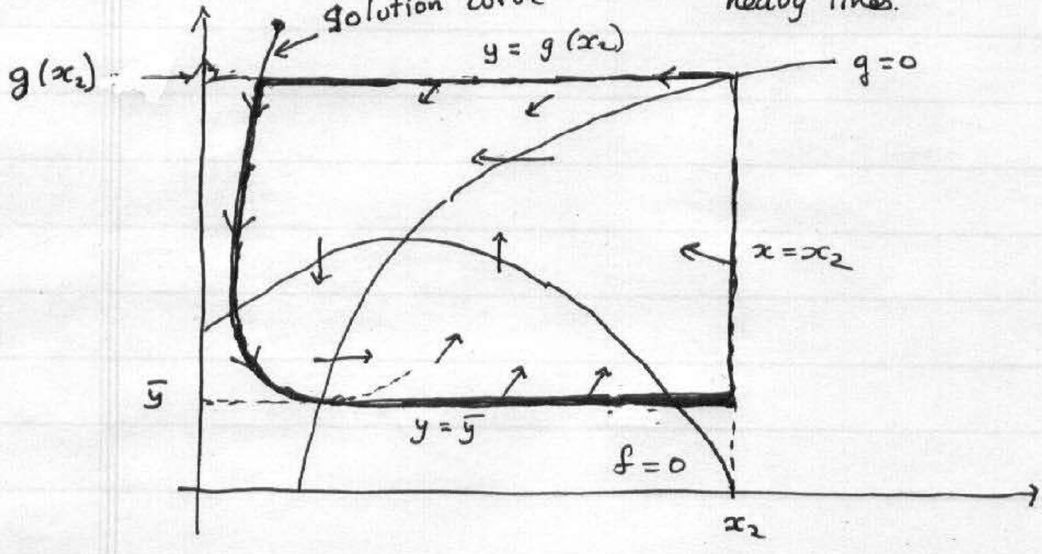


We have discovered that only in the second case is it possible that (x_0, y_0) <u>not</u> be stable.

When the steady state (x_0, y_0) in the interior of $\{x \geq 0, y \geq 0\}$ is ~~not~~ an unstable spiral or unstable mode, the Poincaré - Bondixon theorem implies that a limit cycle exists if there is a bounded region about (x_0, y_0) that is forward invariant. The question then is -- when does such a region exist.

<u>Claim</u> If the line $x = b$ intersects the y-cline $\{g = 0\}$ for some value $b \geq x_2$, then there is a bounded forward invariant region about (x_0, y_0).

The proof is by picture. The invariant region is outlined by the heavy lines.



The left hand side is a piece of a solution from where it intersects $y = g(x_2)$ to where it intersects the y-cline, the rest of the pieces of the boundary are horizontal or vertical lines at the given levels. It is easy to check that the direction field points into or tangent to the boundary.