MATH 336. Notes

Analysis of the Van der Pol oscillator: (PART I)

$$\left.\begin{aligned} \frac{du}{dt} &= v - \left(\tfrac{1}{3}u^3 - u\right) \\ \frac{dv}{dt} &= -u \end{aligned}\right\} \quad (1)$$

This system has exactly one steady state, the origin $(0,0)$. This steady state is an unstable spiral. Our object is to show that (1) has a periodic orbit. We will do so by exhibiting an annular region about the origin that is bounded and that is forward invariant for equation (1). (We say that a region D is forward invariant for the equation $\frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = g(x,y)$ if given a solution $(x(t), y(t))$ and a time t_0 such that $(x(t_0), y(t_0))$ belongs to D, $(x(t), y(t))$ is in D for all future times $t \geq t_0$). Then the Poincaré-Bendixon theorem will imply that the annular region we find will contain a periodic orbit.

We recall that to show a region D is forward invariant for $\frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = g(x,y)$ it suffices to show that the direction field $\langle f(x,y), g(x,y)\rangle$ points into D along its boundary. Recall, also that to check this analytically we can check

$$\vec{n} \cdot \langle f(x,y), g(x,y)\rangle \leq 0 \text{ along the boundary of } D \qquad (2)$$

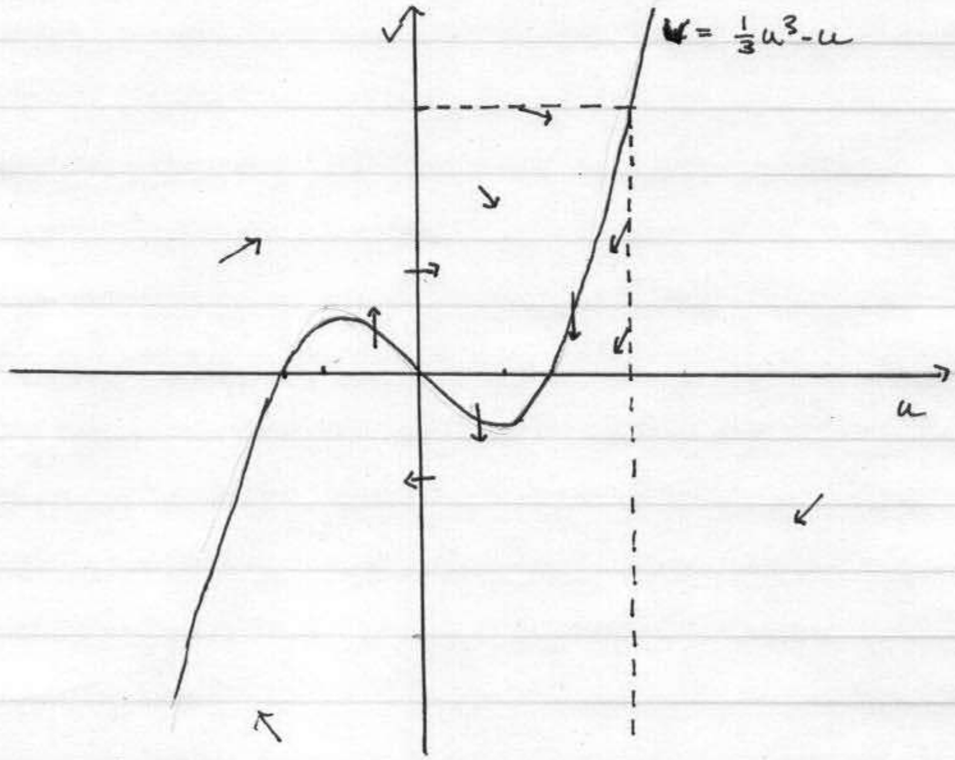
where $\vec{n}$ is an outward pointing vector normal to the boundary.

We shall use the symmetry of (1) with respect to reflection in the origin. Namely, if $(u(t), v(t))$ solves (1) then so does $(-u(t), -v(t))$. This is easy to check: multiply each equation in (1) by -1 and use $-u^3 = (-u^3)$. Then

$$\frac{d}{dt}(-u) = (-v) - \left[\tfrac{1}{3}(-u)^2 - [-u]\right], \quad \frac{d}{dt}(-v) = -(-u).$$

Hence $(-u,-v)$ also solves (1).

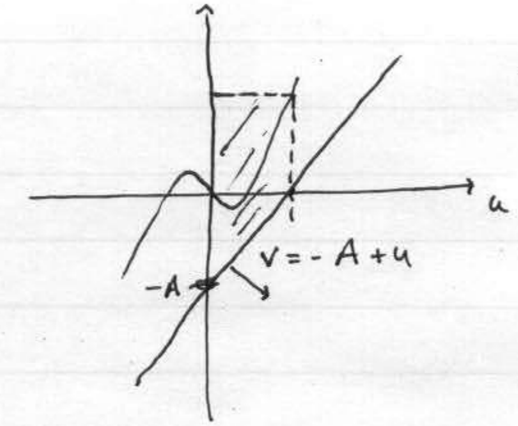
The following figure plots the null clines of (1), which are $v = \frac{1}{3}u^3 - u$ and $u=0$ (the v-axis) and the trend of the direction field in each region defined by the null clines.



The figure also shows a dotted horizontal line segment and a dotted vertical line segment. We will use such line segments for two of the boundaries of our region. Because $\frac{dv}{dt} = -u$ and $u \geq 0$ along the horizontal line segment and because $\frac{du}{dt} = v - G(u) \leq 0$ along the vertical line segment (which is in the region $v \leq G(u)$, the direction field along these segments points into the quadrant below and to the left of these lines.

To continue the construction we need a curve connecting the vertical dotted line to the negative v-axis. A horizontal line won't do -- the direction field in the region $u > 0$ points downward. Instead we will look for a line of the form $v = -A + u$, $A > 0$.

This is a line with slope 1 and you will see in a minute why this is a good choice. Notice that $N = \langle 1, -1 \rangle$ is normal to $v = -A + u$ and points into the half plane below this line. We refer now to criterion (2).



If

$$0 \geq N \cdot \langle v - [\tfrac{1}{3}u^3 - u], -u \rangle$$
$$= \langle 1, -1 \rangle \langle v - [\tfrac{1}{3}u^3 - u], -u \rangle \qquad (3)$$

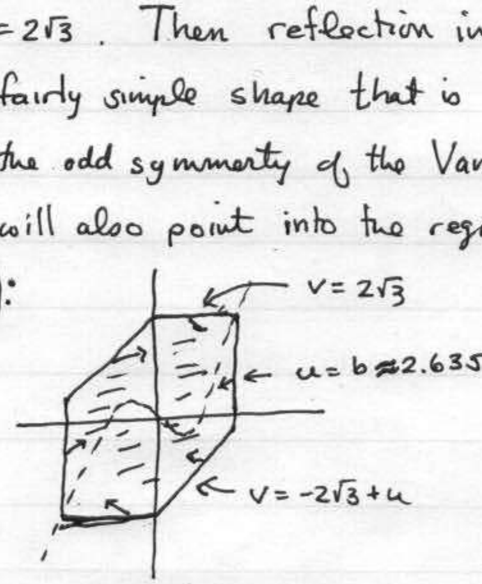
along this line, solutions to (1) can only exit the shaded region in the figure above through the v-axis. Since $v = -A + u$ along the line and since we are only interested in $u \geq 0$, (3) requires

$$0 \geq \langle 1, -1 \rangle \langle -A + u, -[\tfrac{1}{3}u - u], -u \rangle = -A + 3u - \frac{u^3}{3}, \quad \text{for } u \geq 0. \qquad (4)$$

By elementary calculus techniques we find that $\max\limits_{u \geq 0} 3u - \frac{u^3}{3} = 2\sqrt{3}$. Therefore (4) will be true for all $u \geq 0$ if $A = 2\sqrt{3}$. So the line $v = -2\sqrt{3} + u$ will be used for the lower boundary of our region.

Given the lower boundary $v = -2\sqrt{3} + u$, which horizontal and vertical lines should we use? The easiest thing to do is choose the horizontal line at level $v = 2\sqrt{3}$. Then reflection in the origin gives us a nice region of fairly simple shape that is forward invariant (because, by the odd symmetry of the Van der Pol equation, the direction field will also point into the region defined by the symmetric boundary):

The vertical line will be $u = b$ where $\frac{1}{3}b^3 - b = 2\sqrt{3}$; using Maple, $u \approx 2.635$.



By the Poincaré-Bendixon theorem, this region we have constructed must contain a periodic solution, because it is forward invariant and contains only one steady state which is an unstable spiral.

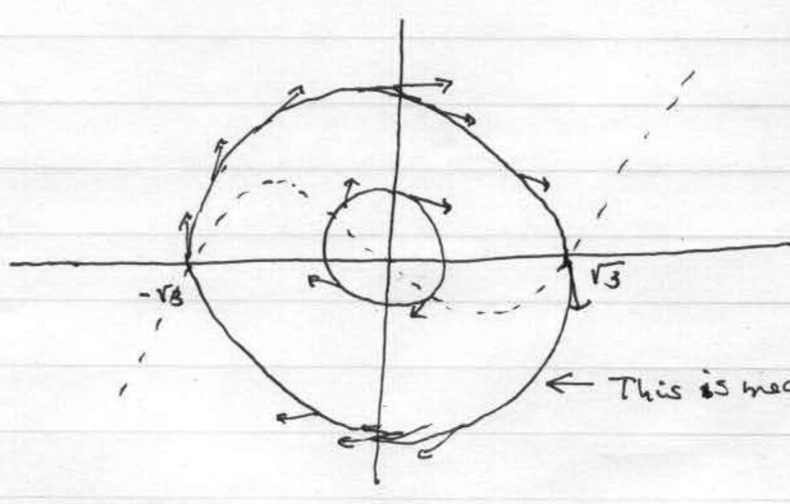
We can improve our analysis by finding a smaller region. Consider the circle $u^2+v^2=r^2$ of radius r centered at the origin. At the point (u,v) on this circle, the vector $\langle -u,-v\rangle$ is normal to (the tangent line to) the circle and points into the circle. Observe that

$$\begin{aligned}\langle -u,-v\rangle \cdot \langle v-[\tfrac{1}{3}u^3-u], -u\rangle \\ &= -uv+\tfrac{1}{3}u^4-u^2+vu \\ &= \tfrac{1}{3}u^2(u^2-3)\end{aligned}$$

Thus

$$\langle -u,-v\rangle \cdot \langle v-\tfrac{1}{3}[u^3-u], -u\rangle \leq 0 \quad \text{so long as} \quad u^2<3$$
$$\text{or equivalently} \quad -\sqrt{3}<u<\sqrt{3}.$$

This implies that along any circle of radius $\sqrt{3}$ or less, about the origin the direction field points into the region external to the disk bounded by that circle. It also implies that as long as a solution $(u(t),v(t))$ stays in the disk $\sqrt{u^2(t)+v^2(t)} \leq 3$ its distance from the origin increases with increasing t.



← This is meant to be a circle!

In conclusion, the region illustrated in the figure below, external to $\{ u^2 + v^2 \leq 3 \}$, but interior to the chevron-shaped forward invariant region described above, is forward invariant. It contains no steady states and hence must contain a periodic orbit which is a limit cycle.