Review questions for Math 351, Midterm #2

1 List all the Euclidean domains that you know, specifying the δ for each.

2 For which of the following properties is it true that if a ring R has the property, then the polynomial ring $R[X]$ has that property?
(a) R is commutative
(b) R is not commutative
(c) R is an integral domain
(d) R is a field
(e) R is a division ring
(f) R is a PID
(g) R is a Euclidean domain.

3 For which of the above properties is it true that if R and S are rings having the property, then $R \times S$ has the property?

4 For which of the following properties is it true that if a ring R has the property and I is any ideal of R such that $I \neq R$, then the quotient ring R/I has the property?
(a) R is commutative
(b) R is an integral domain
(c) R is a field
(d) Every ideal in R is principal.

5 Let $R = \mathbb{Z}[X]$. Show that $R/3R \cong \mathbb{Z}_3[X]$.

6 Let $R = \mathbb{Z}_5[X]$ and $I = (X^3)$. Let $S = R/I$.
(a) How many elements are there in S?
(b) How many units are there in S?
(c) How many zero-divisors are there in S?
(d) True or false: every zero-divisor in S is nilpotent.
(e) How many ideals are there in S?

7 Prove that $Z[i] \cong Z[X]/(X^2 + 1)$.

8 Suppose that $\phi : R \to S$ is a homomorphism of rings. Let I be any ideal of R such that $I \subseteq \ker(\phi)$. Construct a ring homomorphism $\overline{\phi} : R/I \to S$ such that $\mathrm{im}(\overline{\phi}) = \mathrm{im}(\phi)$.

9 What is the result $a = qb+r$ when the division algorithm in $\mathbb{Z}[i]$ is applied with $a = 12+8i$ and $b = 4 + 7i$? (Hint. Check the proof that $\mathbb{Z}[i]$ is Euclidean.)

10 Add the appropriate hypothesis and prove the resulting statement.

If R is a ring, ______________________, and if a, b are nonzero elements of R such that $a \mid b$ and $b \mid a$, then there is $u \in U(R)$ such that $b = ua$.

11 Show that if R is a ring and $\phi : R \to R$ is a ring homomorphism, then $S := \{r \in R \mid \phi(r) = r\}$ is a subring of R. If R is a field, then is S necessarily a field?

12 Suppose that R is a PID, and let $a, b \in R$.

(a) Does there necessarily exist a gcd d of a and b in R? (What does this terminology mean, by definition?)
(b) If d exists, in what sense (if any) is it unique?
(c) If d exists, can d be expressed as an R-linear combination of a and b?
(d) Suppose that $S = R[X]$, or more generally suppose that R is a subring of another integral domain S (not necessarily a PID). Show that d is a gcd of a and b in S. Your answer to (c) should be used somewhere.

13 Let D_5 be the symmetry group of a regular pentagon. How many cyclic subgroups does D_5 have? How many noncyclic subgroups?

14 True or false: If G is a group, then the following cancellation law holds: for any $g, h, x \in G$, $gx = hx$ implies $g = h$.

15 Let G be a group. Show that G is abelian if and only if the mapping $\phi : G \to G$ defined by $\phi(g) = g^2$ is a homomorphism.

16 Let G be a group and $x, y \in G$. Show that if $xy = yx$, then $xy^{-1} = y^{-1}x$.

17 Proof or counterexample: If G is a group, $x, y \in G$, $n \in \mathbb{N}$, and x and y have order n, then $(xy)^n = 1$.

18 Suppose that G is a finite group, and H and K are distinct subgroups of G such that $|H| = |K|$. Show that $H \cup K$ is **not** a subgroup of G, but $H \cap K$ is a subgroup of G.