

Ring	Has Identity	Commutative	Integral Domain	Field	Characteristic	Finite
$\{0\}$	✓	✓			1	✓
\mathbb{Z}	✓	✓	✓		0	
$E = 2\mathbb{Z}$		✓				
\mathbb{Q}	✓	✓	✓	✓	0	
\mathbb{R}	✓	✓	✓	✓	0	
\mathbb{C}	✓	✓	✓	✓	0	
\mathbb{Z}_n	✓	✓	only if n is prime	only if n is prime	n	✓

Other examples:

- Let G be any Abelian group with operation $+$ and identity 0 . Give G the “zero multiplication” defined by $ab = 0$ for all $a, b \in G$. Then G becomes a commutative ring. Every non-zero element of G is a zero divisor. If $G \neq \{0\}$, then G has no (multiplicative) identity.
- Let R_1 and R_2 be rings. Form the product ring $R_1 \times R_2$ by defining: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2)$ for all $x_1, x_2 \in R_1$ and $y_1, y_2 \in R_2$. Notice that $R_1 \times R_2$ is commutative if and only if both R_1 and R_2 are commutative. Also, $R_1 \times R_2$ has an identity if and only if both R_1 and R_2 have identities. One can easily prove that if neither R_1 nor R_2 is a trivial ring, then $R_1 \times R_2$ has zero divisors (so it can never be an integral domain or a field). [Note: If R_1 has m elements and R_2 has n elements, then $R_1 \times R_2$ has mn elements.]
- Let R be a ring. $R^{n \times n}$ is the ring of $n \times n$ matrices with entries in R . Remember if the ij -entries of A and B are a_{ij} and b_{ij} respectively, then the ij -entry of $A + B$ is just $a_{ij} + b_{ij}$ and the ij -entry of AB is $\sum_{k=1}^n a_{ik}b_{kj}$. One can show that $R^{n \times n}$ has an identity if and only if R has an identity. Also, $R^{n \times n}$ is **not** commutative except in the case that (1) R is commutative and $n = 1$ or (2) R has the zero multiplication – that is $ab = 0$ for all $a, b \in R$. [Note: If R has m elements, then $R^{n \times n}$ has m^{n^2} elements.] In our textbook, $M(R) = R^{2 \times 2}$.
- Let R be a ring. $R[[x]] = \{a_0 + a_1x + a_2x^2 + \dots \mid a_i \in R\}$ is the ring of formal power series with coefficients in R . If $f(x) = a_0 + a_1x + \dots$ and $g(x) = b_0 + b_1x + \dots$, then $f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + \dots$ and $f(x)g(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + \dots (\sum_{i=0}^k a_i b_{k-i})x^k + \dots$. It is easy to show that $R[[x]]$ is commutative if and only if R is commutative. Also, $R[[x]]$ has a multiplicative identity if and only if R has 1. In fact, R is an integral domain if and only if $R[[x]]$ is an integral domain.
- Let R be a ring. $R[x] = \{a_0 + a_1x + \dots + a_nx^n \mid a_i \in R\}$ is a subring of $R[[x]]$ called the ring of polynomials with coefficients in R . Again, it is easy to show that $R[x]$ is commutative if and only if R is commutative; $R[x]$ has a multiplicative identity if and only if R has 1; and R is an integral domain if and only if $R[x]$ is an integral domain.