1. Let p be a positive prime integer. Consider the integral domain $\mathbb{Z}[\sqrt{-p}]$.
   (a) Define $N(a+b\sqrt{-p}) = (a+b\sqrt{-p})(\overline{a+b\sqrt{-p}}) = a^2+pb^2$. Show $N(xy) = N(x)N(y)$ for all $x, y \in \mathbb{Z}[\sqrt{-p}]$. Then use that to show $u \in U(\mathbb{Z}[\sqrt{-p}])$ iff $N(u) = 1$.
   (b) Show $\mathbb{Z}[\sqrt{-p}]$ has the ascending chain condition for principal ideals. *Hint:* Notice that if x divides y, then $N(x) \leq N(y)$.
   (c) Even though $\mathbb{Z}[\sqrt{-3}]$ has the ACC (by part (b)), it is not a UFD. Why?
2. (Trick Question) Let R be a PID and let $p \in R$ such that $p \neq 0$. Suppose that $R/(p)$ is an integral domain. What is the field of quotients of $R/(p)$?
3. Let R be a commutative ring with 1. Show that R is an integral domain iff $\{0\}$ is a prime ideal.
4. Let R be an integral domain. Show if $p \in R$ is prime, then (p) is a prime ideal. Is the converse true?
5. Are the rings $\mathbb{Q}[x]/(x^2+1)$ and $\mathbb{Q}[i]$ isomorphic? Either prove it or show why not.
6. Are the rings $\mathbb{Z}_2[x]/(x^2+x+1)$ and $\mathbb{Z}[x]/(x^2+1)$ isomorphic? Either prove it or show why not.
7. Construct a field with 25 elements. *Hint:* This should involve $\mathbb{Z}_5$ and quotients.
8. Let p be a prime integer.
   (a) Let $f(x) \in \mathbb{Q}[x]$. Show $c = a+b\sqrt{p} \in \mathbb{Q}[\sqrt{p}]$ is a root of $f(x)$ iff $\overline{c} = a-b\sqrt{p}$ is a root of $f(x)$.
   (b) Show that $\mathbb{Q}[x]/(x^2-p)$ is isomorphic to $\mathbb{Q}[\sqrt{p}]$.
9. Let $I = (x^2+3x+2)$. Is x^2+1+I a unit in $\mathbb{Q}[x]/I$? Why or why not? If it is a unit find its inverse. Notice that $\mathbb{Q}[x]/I$ is not a field (Why?). Find a zero divisor. Is I a prime ideal? A maximal ideal?
10. Let I and J be ideals of a ring R. Prove that $I+J = \{i+j \mid i \in I \text{ and } j \in J\}$ is an ideal of R.
11. Let $\phi : \mathbb{Z}_6 \to \mathbb{Z}_{25}$ be defined by $\phi(k) = k$. Notice that ϕ is injective and maps onto the set $S = \{0, 1, 2, \ldots, 5\}$. Therefore, S is a subring of $\mathbb{Z}_{25}$ (the image of a ring under a homomorphism is a subring of the codomain). BUT WAIT! $2 \cdot 5 = 10$ in $\mathbb{Z}_{25}$, $2, 5 \in S$, and $10 \notin S$. What's wrong!?!
12. Construct a field with four elements. This field has a subfield isomorphic to $\mathbb{Z}_2$ (why?). Find *all* of the roots of $y^3+1 \in \mathbb{Z}_2[y]$ in your field of four elements.
13. Let R be a Euclidean Domain with "size" function $\delta(x)$. For any fixed $k, \ell \in \mathbb{Z}_{>0}$, show that R is still a Euclidean domain if we use $\delta'(x) = k\delta(x)+\ell$. What goes wrong if $k = 0$?
14. Prove that the field of fractions of $\mathbb{Z}[\sqrt{-p}]$ is $\mathbb{Q}[\sqrt{-p}]$ for any positive prime integer p.
15. Find a homomorphism with domain $\mathbb{Q}[x]$ whose kernel is $(x^5+6x^3+12x+2)$.
16. Prove that (x^2+1) is prime but not maximal in $\mathbb{Z}[x]$. Can we have a prime ideal which is not maximal in a PID?
17. Let R be a ring with 1 such that $a^2 = a$ for all $a \in R$. Let I be a prime ideal of R. Show that R/I has exactly two elements (i.e. it is isomorphic to $\mathbb{Z}_2$).
18. DO YOUR HOMEWORK!!!!!