Here is a long list of practice problems for the final exam. Well, ok they just cover group theory. If you want practice involving rings – look at your old exams and old practice problem lists.

1. Find a subgroup of $S_4$ which is isomorphic to $D_4$. Is the subgroup that you found a normal subgroup?
2. Prove that $|A_n|$ is $n!/2$.
3. Show that $Z_{10}/N$ where $N = \{0, 5\}$ is (group) isomorphic to $Z_5$.
4. Show $\mathrm{SL}_2(\mathbb{Z}_4)$ is a normal subgroup of $\mathrm{GL}_2(\mathbb{Z}_4)$.
5. Recall the quaternion group $Q = \{\pm1, \pm i, \pm j, \pm k\}$ (from problem 14 in section 7.1). Is $Q$ isomorphic to $D_4$?
6. Let $G$ and $H$ be groups. Show that $G \times H$ is abelian if and only if both $G$ and $H$ are abelian. Is it true that $G \times H$ is cyclic if and only if $G$ and $H$ are cyclic?
7. Let $H$ be a subgroup of $G$ where $|G| = 12$. What could $|H|$ be? What are the possible orders of elements of $G$? Must $G$ have an element of order 4?
8. Let $G$ be a group. Prove that $|g| = |g^{-1}|$ for all element $g \in G$. Also, show that $g^{-1} = g^{n-1}$ if $|g| = n$.
9. Is $\mathbb{Q}$ (group) isomorphic to $\mathbb{Z}$?
10. Find all of the subgroups of $S_3$. Which ones are normal? Do the same for $\mathbb{Z}_{12}$.
11. Can a cyclic group have exactly one generator?
12. Let $G$ be a group of order $n > 1$. Explain why $\mathrm{Aut}(G)$ must be isomorphic to a subgroup of $S_{n-1}$. Also, show that the automorphism group of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is ismorphic to $S_3$. How many inner automorphisms does $\mathbb{Z}_2 \times \mathbb{Z}_2$ have?
13. Show that $U_n = U(\mathbb{Z}_n)$ has even order for $n > 2$. Hint: Does $U_n$ have an element of order 2? What are the solutions of $x^2 = 1$ in $\mathbb{Z}_n$?
14. Let $\varphi : G \to H$ be a group homomorphism, and let $g \in G$ have finite order. Show that the order of $\varphi(g)$ divides the order of $g$. Moreover, show that if $|g| = |\varphi(g)|$ for all $g \in G$, then $\varphi$ must be injective.
15. Let $p$ be a prime which divides $n$ the order of a group $G$. Suppose that $g, h \in G$ such that $|g| = |h| = p$. Show that either $\langle g\rangle = \langle h\rangle$ or $\langle g\rangle \cap \langle h\rangle = \{e\}$. Use this to conclude that if there a $k$ elements of order $p$ in $G$, then $p - 1$ divides $k$. Finally, show that any group of order 10 must have an element of order 2.
16. Show that $Z(G)$ (the center of the group $G$) is a normal subgroup of $G$.
17. Can $H = \{(1), (12)\}$ be the kernel of a homomorphism from $S_3$ to some other group $G$?
18. Show that $N = \langle(1234)\rangle$ is *not* a normal subgroup of $S_4$.
19. (Schur's Lemma – Kind of) Show that for a simple group $G$, any homomorphism $\varphi : G \to H$ is either injective or is trivial (i.e. $\varphi(G) = \{e\}$).
20. Show that $N = \{(1), (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of $A_4$. What is $A_4/N$?