Math 351	Workshop #9	November 27, 2006

Group Theory – Thus you must work in groups.

1. Ordered problems. Let G be a group.
   (a) Show that $|a| = |a^{-1}|$ for all $a \in G$.
   (b) Suppose that G is abelian. Show that $|ab|$ divides $\operatorname{lcm}(|a|, |b|)$.
   Note 1: Notice that in $\mathbb{Z}_{10}$, $|1| = 10$ and $|4| = 5$ but $|1 + 4| = |5| = 2 \neq \operatorname{lcm}(10, 5) = 10$. However, 2 does divide 10.
   Note 2: If G is not abelian, then the conclusion in part (b) does not necessarily hold (see section 7.2 problem #18).

2. Cyclic Groups.
   (a) Let $G = \langle a \rangle$ be the cyclic group generated by a where $|a| = \infty$. Find the order of each element of G.
   (b) Redo part (a) for the cases where $|a|$ is 4, 6, 8, and 12.
   (c) Let $G = \langle a \rangle$ be the cyclic group generated by a where $|a| = n < \infty$. Find a formula for the order of each element a^k of G. Prove your formula.

3. Subgroups anyone?
   (a) Let R be a commutative ring with 1. Show that $\mathrm{GL}_2(R) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, \det(A) = ad - bc \in U(R) \right\}$ is a group under matrix multiplication. This group is called the "general linear group" (of 2 by 2 matrices with entries in R). *Hint:* Remember $A^{-1} = (\det(A))^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
   (b) Let $\mathrm{SL}_2(R) = \{A \in \mathrm{GL}_2(R) \mid \det(A) = 1\}$. Show that $\mathrm{SL}_2(R)$ is a subgroup of $\mathrm{GL}_2(R)$. This group is called the "special linear group" (of 2 by 2 matrices with entries in R).
   (c) How many elements are there in $\mathrm{GL}_2(\mathbb{Z}_3)$? How many elements in $\mathrm{SL}_2(\mathbb{Z}_3)$?

4. Let G be a cyclic group of order n. Show that there is a unique subgroup of order k for each divisor k of n. *Hint:* What is the order of $a^{n/k}$ if $|a| = n$?

5. Permutations
   (a) Write each element of S_4 in cycle notation.
   (b) Which permutations in S_4 are even (i.e. what are the elements of A_4)? Which ones are odd?
   (c) Find the order of each element of S_4.
   (d) Determine $Z(S_4)$ (the center of S_4).
   (e) Find $Z(S_n)$ for all $n \geq 1$. (Be careful when dealing with $n = 2$!)