Let the set of integers be denoted by $\mathbb{Z}$. A Gaussian integer is a complex number of the form $x = a + bi$, where $a, b \in \mathbb{Z}$ and $i^2 = -1$. We write $\mathbb{Z}[i]$ for the set of Gaussian integers. In this Workshop, all variables in the range $q$–$z$ are Gaussian integers. Here is the Division Algorithm: Let $x, y$ be nonzero Gaussian integers. Then there exist $q, r$ in $\mathbb{Z}[i]$ with $y = qx + r$ and $|r| < |x|$. (The length of a Gaussian integer $x$ is $|x| = \sqrt{a^2 + b^2}$, and $|xy| = |x||y|$.)

Say $x$ divides $y$ (and write $x|y$) if $y = qx$ for some $q$ in $\mathbb{Z}[i]$. For example, $(2 + i)|5$ because $5 = (2 + i)(2 - i)$. A common divisor of $y, z$ is an $x$ so that $x|y$ and $x|z$. We call $x$ a greatest common divisor of $y, z$ if $x$ is a common divisor and if $t$ is any other common divisor then $|t| < |x|$.

1. a) If $y \neq 0$, show that $y$ has only finitely many divisors.
   b) If $x|y$ show that $i, -i, -x, ix, -ix$ are also divisors of $y$. (And so are $\pm 1, \pm i$.)

A Gaussian integer $x$ is said to be prime if $|x| > 1$ and the only divisors of $x$ are $\pm 1, \pm i, \pm x, \pm ix$.

   c) If $|x| = \sqrt{p}$ for some prime integer $p$, show that $x$ is a prime in $\mathbb{Z}[i]$. (This shows that $2 + i, 2 - i$ are primes.)

Fix nonzero $y, z$ and let $I$ denote the set of all Gaussian integers of the form $t = uy + vz$, where $u, v$ are Gaussian integers. For example, if $y = 2 + i$ and $z = 2 - i$ then $I$ contains the Gaussian integers $2 = (-i)(2 + i) + (i)(2 - i)$ and $5 = (2 + i)(2 - i)$.

2. a) If $x$ is any common divisor of $y, z$, show that $x|t$ for every $t \in I$.
   b) Show that there is a nonzero element $x$ in $I$ with $|x| \leq |t|$ for every nonzero $t$ in $I$.
   c) Show that the $x$ in (b) is a common divisor of $y, z$.
   d) Show that the $x$ in (b) is a greatest common divisor of $y, z$.

3. a) If $x$ is a prime Gaussian integer and $x|yz$, show that either $x|y$ or $x|z$.
   b) If $x$ is prime, $x|z$ and $z = y_1 y_2 \ldots y_n$, show that $x$ divides some $y_i$. 