1. Let \( V \) be a two-dimensional vector space over \( \mathbb{R} \) with basis \( \{ v, w \} \). Fix a real number \( r \) and define the operation \( \cdot \) on \( V \) by

\[
(a_1 v + b_1 w) \cdot (a_2 v + b_2 w) = (a_1 a_2 + rb_1 b_2)v + (a_1 b_2 + a_2 b_1)w
\]

for each \( a_1, a_2, b_1, b_2 \in \mathbb{R} \).

a) Show that \( V \), with + being vector space addition and product \( \cdot \), is a commutative ring with identity element \( v \).

b) If \( r < 0 \), show that \( V \) is isomorphic to \( \mathbb{C} \). (Hint: what is \( w \cdot w \)? How about \((w/\sqrt{|r|})^2\)?)

c) If \( r > 0 \), show that there is a homomorphism \( f_+: V \to \mathbb{R} \) sending \( w \) to \( \sqrt{r} \), and a second homomorphism \( f_-: V \to \mathbb{R} \) sending \( w \) to \(-\sqrt{r}\).

d) If \( r > 0 \), use the \( f_\pm \) of part (c) to construct a homomorphism \( f: V \to \mathbb{R} \times \mathbb{R} \). Then prove that \( f \) is an isomorphism.

2. Let \( C^0 \) denote the set of functions \( f: \mathbb{R} \to \mathbb{R} \) which are continuous at \( t = 0 \), that is \( \lim_{t \to 0} f(t) \) exists and equals \( f(0) \). It is shown in the book that \( C^0 \) is a commutative ring with identity. For integers \( n > 0 \), let \( C^n \) denote the set of (continuous) functions in \( C^0 \) such that the first \( n \) derivatives \( f', f'', \ldots, f^{(n)} \) are all defined and continuous at \( t = 0 \). They are distinct sets; for example, in Calculus we show that the function \( t \to |t| \) is in \( C^0 \) but not \( C^1 \), and \( t \to t|t| \) is in \( C^1 \) but not in \( C^2 \).

a) Show that each \( C^n \) is a subring of \( C^0 \), with \( C^n \subset \cdots \subset C^2 \subset C^1 \subset C^0 \).

b) Let \( V \) be the ring of problem 1 with \( r = 0 \). Show that the function \( T: C^1 \to V \) defined by \( T(f) = f(0)v + f'(0)w \) is a ring homomorphism.

c) Is the homomorphism \( T \) in b) one-to-one? Is it surjective?