1. All you know is that \( au + bv = 6 \) for some integers \( a, b, u, v \). What could \( (a, b) \) be? Also, is it possible that \( u \) and \( v \) are relatively prime?

2. Let \( a, b \in \mathbb{Z} \). Suppose that \( a \) divides \( b \) and \( b \) divides \( a \). Is it necessarily true that \( a = b \)? What can you say if \( f(x) \) divides \( g(x) \) and \( g(x) \) divides \( f(x) \) when \( f(x), g(x) \in \mathbb{F}[x] \) for some field \( \mathbb{F} \)?

3. Let \( p \) be a positive prime integer. Show \( \sqrt{p} \) is irrational.

4. Does \( 2000x \equiv 4 \mod 19875 \) have any solutions? **No serious calculations needed!!!**

5. Write out a multiplication table for \( \mathbb{Z}_3 \times \mathbb{Z}_2 \). Find all units and zero divisors.

6. Let \( U = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \mid a, b, c, d, e, f \in \mathbb{R} \right\} \) be the set of 3x3 real upper-triangular matrices. Show that \( U \) is a subring of \( \mathbb{R}^{3 \times 3} \).

7. Let \( R \) be a ring and define \( Z(R) = \{ r \in R \mid ar = ra \text{ for all } a \in R \} \) to be the center of \( R \). Show that \( Z(R) \) is a subring of \( R \). What is \( Z(\mathbb{Z}) \)? What is \( Z(M(\mathbb{R})) \)?

8. Show that \( U(R_1 \times R_2) = U(R_1) \times U(R_2) \) for any two rings with identity \( R_1 \) and \( R_2 \).

9. Let \( R \) be a commutative ring with 1, and \( r \in R \). Define \( L_r : R \rightarrow R \) by \( L_r(x) = rx \) for all \( x \in R \).

   Prove that \( L_r \) is injective iff \( r \) is a nonzero nonzero divisor. Prove that \( L_r \) is surjective iff \( r \) is a unit.

10. Let \( a, b \in R \) (a ring). Show that \( -ab = (-a)b = a(-b) \) and \( -(a) = a \) just using ring axioms.

    Use your results above to show that \( (-1)(-1) = 1 \) if \( R \) has a multiplicative identity 1. One more thing...show that 3(\( ab \)) = 3(ab) = a(3b).

11. Let \( R \) and \( S \) be rings. Prove that \( R \times S \) is isomorphic to \( S \times R \). Let \( R \) be an integral domain. Prove that \( R \times R \) is not isomorphic to \( R \).

12. Let \( a, b, c \in \mathbb{Z} \). Show that if \( c \) divides \( b \) and \( (a, b) = 1 \), then \( (a, c) = 1 \). Now prove the same result for polynomials with field coefficients.

13. Is \( A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \) a unit in \( M(\mathbb{Z}) \)? Why or why not? Is \( A \) a unit in \( M(\mathbb{Z}_7) \)?

14. Factor \( x^4 - 9 \) in \( \mathbb{Q}[x], \mathbb{R}[x], \) and \( \mathbb{C}[x] \). Also, factor \( x^4 - 9 \) in \( \mathbb{Z}_2[x] \).

15. Let \( \mathbb{F} \) be a subfield of \( \mathbb{C} \), \( \phi : \mathbb{F} \rightarrow \mathbb{F} \) be an automorphism of \( \mathbb{F} \) such that \( \phi(c) = c \) for all \( c \in \mathbb{Q} \) (\( \phi \) fixes the rationals), and let \( f(x) \in \mathbb{Q}[x] \). Show that \( r \in \mathbb{F} \) is a root of \( f(x) \) iff \( \phi(r) \) is a root of \( f(x) \).

16. Prove that \( \mathbb{Q}[^2] = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \} \) is a subfield of \( \mathbb{C} \). Also, prove that \( \phi : \mathbb{Q}[\sqrt{2}] \rightarrow \mathbb{Q}[\sqrt{2}] \) defined by \( \phi(a + b\sqrt{2}) = a - b\sqrt{2} \) is an automorphism of \( \mathbb{Q}[\sqrt{2}] \). In fact, using the previous problem, one can show that the only automorphisms of \( \mathbb{Q}[\sqrt{2}] \) are \( \phi \) and the identity map.

17. Let \( f(x) \in \mathbb{F}[x] \) (where \( \mathbb{F} \) is a field) be a polynomial of degree 5. Suppose that \( f(x) \) has no roots in \( \mathbb{F} \) and no quadratic factors (no polynomial of degree 2 divides \( f(x) \)). Can I then conclude that \( f(x) \) is irreducible? Why or why not?