

Math 351:03 — Fall 1999
MW4 SEC-217
Prof. Bumby

Comments on problems in Workshop 1

Problem 1 was selected for write-up. Individual comments will be given in judging your written presentation.

Problem 2 was discussed during the workshop class. Here is an extension of that discussion. First, we quote the statement to make it easier to refer to it.

Suppose we have mappings $f: S \rightarrow T$ and $g: T \rightarrow S$ such that $fg = i_T$. Show that f is onto and g is one-to-one. Give an example in which neither f nor g is an equivalence. What is $gf: S \rightarrow S$ in your example?

Imagine what a fully detailed proof would look like. It would begin by introducing f , g , S , and T to reserve those names to refer only to those things mentioned in the statement. This opening statement is important because it allows the elements of S and T to be used as variables later in the proof.

The proof that f is onto is likely to be independent of the proof that g is one-to-one, so one will appear in full; then a statement will be made to clear away any conventions or local definitions needed for that proof to allow the second proof full freedom to supplement what was fixed in the opening statement. The second proof is then given; followed by a sign that the proof is finished.

We look at the two parts. Since they are independent, it doesn't matter which we look at first. Indeed, it will be clearer if we try to keep them completely separate. Each of the statements we are proving translates into a formal statement of *first order logic*. That is, in addition to the familiar proposition connectives like **and**, **or**, **not**, **implies**, we can use **quantifiers** like *for all* or *there exist* in the context of elements of a fixed set (such as S or T in this problem). The verbal description and the formal statement are synonymous, but the formal statement determines the structure of the proof. Indeed, this is often the only way that quantifiers appear when we describe the proof. To prove f is onto, we need to know what that means as a formal mathematical statement. Here it is:

- For all $t \in T$,
- there exists $s \in S$ such that
- $fs = t$.

Any proof of this must have the form:

- Given $t \in T$,
- let $s = \underline{\hspace{2cm}}$ (a definition that identifies a particular element of S)

The expression fs is then computed from the given information and shown to be equal to t . This computation may be routine; so that all the work in the proof consists of filling in one blank. In this case, we have so little to work with that the result can be true in this generality only if it has an easy proof. The only way we have to produce elements of S is by applying the function g to elements of T , and the only element of T we have is t . We are lucky: if $s = gt$, then

$$fs = f(gt) = (fg)t = i_T t = t.$$

Note the use of parentheses in the intermediate expressions to clarify the reason for each equality. (In case it isn't completely obvious, the first is the definition of s , the second is the definition of composition of mappings, the third is a stated hypothesis, and the fourth is the definition of i_T .) Apart from the structure forced by the quantifiers, this chain of equalities is the whole proof.

To prove g is one-to-one, we need to know what that means as a formal mathematical statement. Here it is:

- For all $t_0 \in T$ and
- for all $t_1 \in T$,
- if $gt_0 = gt_1$, then
- $t_0 = t_1$.

This is the kind of statement that only a mathematician could love. We introduce two different names t_0 and t_1 for something that we aim to prove is the same element of T . This keeps the interpretation of what is being proved from getting in the way of the formal rules that restrict what can appear in a proof.

Any proof of this must have the very simple form:

- Given t_0 and t_1 in T ,
- suppose that $gt_0 = gt_1$

followed by an argument ending with

- $t_0 = t_1$.

Since there are no blanks to fill in, the proof of this can only be algebraic manipulation using the definitions and hypotheses. It can also be stated as a chain of equalities:

$$t_0 = i_T t_0 = (fg)t_0 = f(gt_0) = f(gt_1) = (fg)t_1 = i_T t_1 = t_1,$$

or we could work out from the middle, writing the sequence of statements

$$\begin{aligned} gt_0 &= gt_1 \\ f(gt_0) &= f(gt_1) \\ (fg)t_0 &= (fg)t_1 \\ i_T t_0 &= i_T t_1 \\ t_0 &= t_1 \end{aligned}$$

Here we begin with the statement that is the hypothesis of the implication we need to prove. The next line applies the mapping f , using very strongly the idea that the value of a function depends only on the element to which it is applied and not the name of the element. The definition of composition is then invoked, followed by the hypothesis that $fg = i_T$ and the defining property of i_T .

It remains to give an example. It would be nice if we could make the example simple, so we look for finite sets S and T . Since each appears as the codomain of a mapping, and since the empty set can appear in this position only if the domain of the mapping is also empty, we need both S and T to be nonempty. Using the conclusion of the result we proved, g will be one-to-one, which means that g uses the set T to count some of the elements in S . If we are to avoid having the sets equivalent, there must be some other points in S . This suggests that we should take T to be a set with one point, say $\{0\}$ and S a set with two points, say $\{a, b\}$. Using symbols that look different allows us to instantly check the syntax of statements about the examples, since each symbol can appear in only one context. We are now forced to define f so that $fa = fb = 0$, while there are two choices for g . We pick one: $g0 = a$. Formal calculation shows that $fg0 = fa = 0$; this completely defines fg , and verifies that $fg = i_T$. Similarly $gfa = g0 = a$ and $gfb = g0 = a$; this completely defines gf , and verifies that $gf \neq i_T$.

Problem 3 also shows the benefit of distinguishing things to allow a simple proof of a general theorem that will also apply when these things are two names for the same object. If $e: S \rightarrow T$ is a bijection, there is a general construction that translates statements about S to statements about T and *vice versa*. This problem examines how that can be used to relate $A(S)$ to $A(T)$. This is so important that we will return to it later. For now, let us make explicit some of the things that are hidden in the statement of the problem, but must appear in its proof.

The opening statement of a solution to this problem will fix notation for the sets S and T , the mappings $e: S \rightarrow T$ and $e^{-1}: T \rightarrow S$ (this name is used to emphasize the e is a bijection), and the mappings $\phi: A(S) \rightarrow A(T)$ and $\beta: A(T) \rightarrow A(S)$ introduced in the statement of the problem. Since formulas are given for ϕ and β , the verification that they are *mappings* is easy, and may be considered part of the background of the problem.

The problem seeks to prove that

$$\phi \circ \beta = i_{A(T)} \text{ and } \beta \circ \phi = i_{A(S)}.$$

Before we try to show that, we need to be sure that they **can** be equal. That is, that they are the same kind of objects. To do this, we identify the set in which equality will be tested. For example, $\phi \circ \beta$ must be the result of applying ϕ to the result of applying β to something. The definition of β shows that it applies to elements of $A(T)$ and gives elements of $A(S)$. Since the result is in $A(S)$, we are sure that ϕ will apply to it and that the result will belong to $A(T)$. This shows that $\phi \circ \beta$, like $i_{A(T)}$ is the name of an element of $A(A(T))$.

In order to test the equality of elements of $A(A(T))$, one must show that they have the same values at all elements of $A(T)$. This requires introducing a new symbol for a general elements of $A(T)$. Note that the symbol g was used for exactly this purpose when we were defining β , so it is a candidate to be used for that purpose here. This symbol is introduced, and the elements of $A(A(T))$ we are interested in are applied to it, giving expressions in $A(T)$, described in terms of g that must be equal for all g .

Now we must test equality in $A(T)$. To do this, we apply these functions to a general element of T . Such an element must be introduced and given a name — the symbol t is a common choice.

In each case, equality of functions is expressed as taking the same value at all points of the domain of the function. When the proof is written, this universal quantifier is usually expressed by a phrase, “let $g \in A(T)$ ”, or “let $t \in T$ ”, at the beginning of the proof of the formula. When the resulting general instance of the formula has been proved, we consider that the equality of functions has been established, and may say so.

In each case, even the *general* instance of the desired formula may be transformed in ways that have not yet been established as algebraic operations on the functions.

One notational consideration should be mentioned. Wherever possible, I believe that parentheses should be omitted when we have statements that resemble the associative law. By contrast, when dealing with mappings between spaces of functions, it appears necessary to *require* parentheses. We could write “the function $\beta(g)$ ” as (βg) where the outer parentheses serve as a reminder that this operation needs to be done before this function can be evaluated at an element of S .

Problem 4 will be restated in Workshop 2.

End
