Math 351:03 — Fall 1999
MW4 SEC-217
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Workshop 1, Textbook Sections 1.1 thru 1.5

1. Let $\mathbb{N}$ denote the set of nonnegative integers, and $\mathbb{N}_+$ the set of positive integers. We will give two mappings that have attracted a lot of attention. However, this course concentrates on mappings that are one-to-one and onto. In order to see if they belong in this course, you should ask: (a) is the mapping one-to-one? (b) is the mapping onto? What are the answers for the functions

(I) $A\colon \mathbb{N}_+ \to \mathbb{N}$ is defined by letting $A(n)$ be the sum of all divisors of n that are smaller than n. Thus, $A(1) = 0$ since it is an empty sum, and $A(6) = 1 + 2 + 3 = 6$.

(II) $C\colon \mathbb{N}_+ \to \mathbb{N}_+$ is defined by

$$C(n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}$$

2. Suppose we have mappings $f\colon S \to T$ and $g\colon T \to S$ such that $fg = i_T$. Show that f is onto and g is one-to-one. Give an example in which neither f nor g is an equivalence. What is $gf\colon S \to S$ in your example?

3. Suppose $e\colon S \to T$ is a bijection (i.e., one-to-one and onto), so that the discussion preceding Lemma 1.3.4 shows that there is a bijection $e^{-1}\colon T \to S$, and Lemma 1.3.4 itself shows that $e \circ e^{-1} = i_T$ and $e^{-1} \circ e = i_S$. If $f \in A(S)$, define $\phi(f) \in A(T)$ by

$$\phi(f) = e \circ f \circ e^{-1};$$

and if $g \in A(T)$, define $\beta(g) \in A(S)$ by

$$\beta(g) = e^{-1} \circ g \circ e.$$

This gives $\phi\colon A(S) \to A(T)$ and $\beta\colon A(T) \to A(S)$. Show that

$$\phi \circ \beta = i_{A(T)} \text{ and } \beta \circ \phi = i_{A(S)}.$$

Note: together with problem 2, this establishes a correspondence between $A(S)$ and $A(T)$ when S is used to name the elements of T.

4. Let $S = \{0, 1, 2, 3\}$, and model S_4 by $A(S)$. Let i be the identity, given by $in = n$ for $n \in S$; let f be defined by

$$f0 = 1,\ f1 = 2,\ f2 = 3,\ f3 = 0;$$

and let g be defined by

$$g0 = 1,\ g1 = 0,\ g2 = 2,\ g3 = 3.$$

Find the powers of f, and show that $f^4 = i$. Show that $g^2 = i$. Show that all 24 elements of $A(S)$ can be obtained as products of terms, each of which is f or g. (Hint: the easiest way to prove existence of such expressions does not necessarily give the shortest expression for every element.)

End workshop 1