Math 351:03 — Fall 1999
MW4 SEC-217
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Workshop 2, Textbook Sections 1.4 thru 2.4

We begin by restating an earlier problem.

4*. [Old] Let $S = \{0, 1, 2, 3\}$, and model S_4 by $A(S)$. Let i be the identity, given by $in = n$ for $n \in S$; let f be defined by

$$f0 = 1,\ f1 = 2,\ f2 = 3,\ f3 = 0;$$

and let g be defined by

$$g0 = 1,\ g1 = 0,\ g2 = 2,\ g3 = 3.$$

Find the powers of f, and show that $f^4 = i$. [New] Let

$$H = \{\, h \in A(S) : h1 = 1 \,\}.$$

(a) Show that H is a subgroup. Elements of H can be viewed as permutations of $\{1, 2, 3\}$, and all such permutations arise, so H can be considered as S_3. Thus H has order 6.

(b) The proof of Lagrange's Theorem shows that there are four left cosets gH. To aid in characterizing these cosets, show that, if g_0 and g_1 lie in the same coset, then $g_0 0 = g_1 0$.

(c) Show that $f^0 = i$, $f^1 = f$, f^2 and $f^3 = f^{-1}$ lie in different cosets. Since we now have four representatives of four cosets. every element of $A(S)$ lies in one of iH, fH, f^2H, $f^{-1}H$.

(d) Show that $g \in fH$, so that $f^{-1}g \in H$. Also identify the cosets containing each gf^i, $i = 0, 1, 2, 3$ Using the appropriate coset representative, this gives elements of the form $f^j g f^i \in H$.

(e) Our previous study of S_3 (p. 17 of textbook), when applied to H shows that we could obtain all elements of H from two of those found in (d). From this, conclude [old] that all 24 elements of $A(S)$ can be obtained as products of terms, each of which is f or g. (Now, you shouldn't need a hint.)

5. Suppose that z_0 and z_1 are complex numbers and $|z_0| = |z_1| = 1$.

(a) Show that $|z_0 + z_1| \le 2$.

(b) If α is a complex number with $|\alpha| \le 2$, show how to find complex numbers z_0 and z_1 with $|z_0| = |z_1| = 1$ and $z_0 + z_1 = \alpha$.

(c) Since $|-z| = |z|$, there are infinitely many pairs (z_0, z_1) satisfying (b) if $\alpha = 0$; how many solutions are there if $\alpha \ne 0$?

. . . continued on other side

6. (Based on problems in Section 2.2 of textbook). For some fixed integer n, suppose that G is a group such that, for all a and b in G, $(ab)^n = a^n b^n$. The identity element of G will be denoted e, all other letters are available to use as variables — and we will take advantage of that (it is more customary to overwork a few letters, but arguments are easier to follow if the general variables used to express a formula are *never* used to name individuals to which the formula applies).

(a) Show that, for all c and d in G,

$$(cd)^{n-1} = d^{n-1}c^{n-1}$$

(b) Show that, for all f and g in G,

$$f^n g^{n-1} = g^{n-1} f^n$$

(c) Show that, for all h and k in G,

$$\begin{aligned}
(hkh^{-1}k^{-1})^n &= hk^n h^{-1} k^{-n} \\
(hkh^{-1}k^{-1})^n &= h^n k h^{-n} k^{-1} \\
(hkh^{-1}k^{-1})^{n-1} &= kh^{1-n} k^{-1} h^{n-1} \\
(hkh^{-1}k^{-1})^{n-1} &= k^{1-n} h k^{n-1} h^{-1}
\end{aligned}$$

(See hint for problem 6c on page 50 of the textbook).

(d) If p and q are in G, find *many* expressions for

$$(pqp^{-1}q^{-1})^{n(n-1)}$$

(e) Show that, for all p and q in G,

$$(pqp^{-1}q^{-1})^{n(n-1)} = e$$

End workshop 2