Math 351:03 — Fall 1999
MW4 SEC-217
Prof. Bumby

Workshop 3, Textbook Sections 2.1 thru 2.5

6*. (Based on problems in Section 2.2 of textbook). Further properties of groups satisfying $(ab)^n = a^n b^n$. You may need to refer to the previous statement for hints on the present problem. Suppose that all a and b in G satisfy this condition with $n = 3$.

(a) If $n = 3$, the previous part (b) specializes to

$$f^3 g^2 = g^2 f^3 \qquad (A)$$

for all f and g in G. Prove (A).

(b) Let $c \in G$ be an element of order 5, i.e., $c^5 = e$ where e is the identity element of G. Show that $cx^2 = x^2 c$ for all $x \in G$.

(c) For the same c as in (b), show that $cx^3 = x^3 c$ for all $x \in G$.

(d) For the same c as in (b), show that $cx^n = x^n c$ for all $x \in G$ and all $n > 1$.

(e) If G is a finite group, show that for every $x \in G$ there is an integer $n > 1$ such that $x^n = x$.

(f) If G is a finite group, and for the same c as in (b), show that $cx = xc$ for all $x \in G$. That is, $c \in Z(G)$ — the center of G.

7. The symmetries of the cube. Imagine a cube resting on a flat surface. Draw an outline of the square face in which the cube meets the surface. Let the cube be moved in some way and returned to this surface so that its base fills the outline you have drawn. This is what we shall mean by a "rigid motion of the cube". If we only note the effect of this motion, and not the path taken by the cube in returning to its original location, we get a group called the group of symmetries of the cube. This is a finite group, since the symmetry is completely determined by the permutations of faces, edges, and vertices of the cube that it induces. Partial information about these permutations can be used to find some properties of the group.

Supposed the faces of cube are marked with numbers from 1 to 6. (Such a *die* is quite common in the casinos of Atlantic City.) An example of "partial information about a symmetry" is the number on the top face. If the top face originally shows a 1, then **prove that** the symmetries leading to 1 on the top face form a subgroup.

Then, show that there are 6 cosets of this subgroup, corresponding to the number showing on the top face after the symmetry.

Then show that the subgroup has only 4 elements (if only rotations are allowed). This shows that the whole symmetry group has 24 elements.

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8. Construct a nonabelian group G of order 21. General techniques related to this will appear later in chapter 2. The main consequence of these results will be that there must be a normal subgroup H of order 7. Since we are only trying to construct such a group, and not trying to show that this is the only possibility, we can begin by assuming that G will have such a subgroup.

(a) Show that every element not in H has order 3.

(b) Let $h \in H$ with $h \neq e$, and let $k \notin H$. The every element of G can be written in the form $h^m k^n$ with $0 \leq m < 7$ and $0 \leq n < 3$.

(c) Since H is a normal subgroup consisting only of powers of h, $khk^{-1} = h^r$ for some r. What does this say about k^3hk^{-3}?

(d) What must k^3hk^{-3} be? What restriction do this put on r?

(e) It turns out that $r = 2$ is allowed. Give the functions η and κ defined by $\eta(x) = hx$ and $\kappa(x) = kx$ on elements of the form $h^m k^n$ in this case. All that will be needed to show that we have constructed a group is to show that η and κ satisfy $\eta^7 = \kappa^3 = i$ and $\kappa\eta = \eta^2\kappa$. Verify these properties of your functions.

End workshop 3