

# Math 351:03 — Fall 1999

## MW4 SEC-217

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### Workshop 4, Textbook Sections 2.4 thru 2.7

**3\***. The original form of this problem said that, when  $e: S \rightarrow T$  is a bijection, it led to the construction of a bijection between  $A(S)$  and  $A(T)$ . In detail, this was given by mappings  $\phi: A(S) \rightarrow A(T)$  and  $\beta: A(T) \rightarrow A(S)$  defined by

$$\phi(f) = e \circ f \circ e^{-1}$$

$$\beta(g) = e^{-1} \circ g \circ e$$

for  $f \in A(S)$  and  $g \in A(T)$  (correcting a misprint in the original statement).

(a) [original problem 3] Show  $\beta \circ \phi = i_{A(S)}$ . Note that the definition of equality of functions means that this statement is *the same* as  $(\beta \circ \phi)(f) = f$  for  $f \in A(S)$ . Use the definition of composition of mappings and the definitions of  $\beta$  and  $\phi$  to see what the quantity on the left is as an element of  $A(S)$ . The definitions of  $\beta$  and  $\phi$  used only composition of mappings, so result should be clear without examining the behavior of  $(\beta \circ \phi)(f)$  and  $f$  at individual elements of  $S$ .

(b) [new] If  $T = S$ , the mapping  $e: S \rightarrow T$  can now be considered as an element of  $A(S)$ , so that  $\beta$  and  $\phi$  map  $A(S)$  to itself. Show that  $\beta$  is a homomorphism.

(c) [new] Example 9 of Section 2.5 claims to prove a form of (b) for a general group  $G$  and calls  $\beta$  the *inner automorphism induced by  $e$* . Obtain this as a corollary of Cayley's theorem by showing that, if  $G$  is a subgroup of  $A(S)$  and  $e \in G$ , then  $\beta(G) \subseteq G$ . (A similar proof will show the same for  $\phi$ , which will show that these mappings are isomorphisms.)

**9**. This problem will describe the action of  $S_4$  on the set of *pairs* of distinct elements of  $S = \{0, 1, 2, 3\}$ .

(a) Let  $A = \{0, 1\}$ ,  $B = \{0, 2\}$ ,  $C = \{0, 3\}$ ,  $D = \{1, 2\}$ ,  $E = \{1, 3\}$ ,  $F = \{2, 3\}$ , and let  $T = \{A, B, C, D, E, F\}$ . Then for  $f \in A(S)$  and  $X \in T$  we define

$$fX = \{fx : x \in X\}.$$

Show that all such  $fX \in T$ ; that is, show that  $fX$  is a subset of  $S$  with exactly two elements, and  $T$  is the set of all such subsets of  $S$ .

(b) Illustrate (a) by tabulating all  $gX$  for  $X \in T$  if  $g$  is the element of  $A(S)$  with  $g0 = 1$ ,  $g1 = 0$ ,  $g2 = 2$ ,  $g3 = 3$ .

(c) Illustrate (a) by tabulating all  $hX$  for  $X \in T$  if  $h$  is the element of  $A(S)$  with  $h0 = 1$ ,  $h1 = 2$ ,  $h2 = 0$ ,  $h3 = 3$ .

(d) Illustrate (a) by tabulating all  $kX$  for  $X \in T$  if  $k$  is the element of  $A(S)$  with  $k0 = 1$ ,  $k1 = 2$ ,  $k2 = 3$ ,  $k3 = 0$ .

We will return to this later, explaining the significance of looking at only 3 of the 24 elements of  $A(S)$ .

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**10.** Introduction to groupoids and the 15-puzzle. About 100 years ago, and from time to time since then, there has been a popular diversion made from a four by four array of square blocks with one removed. The fifteen remaining blocks are numbered from 1 to 15. The space freed by removing a block allows these blocks to be permuted by sliding a block adjacent to the free space (horizontally or vertically, but not diagonally) into that space. Since the moves of this puzzle involve permutations, one suspects that there is some group theory involved. However, a more general formulation appears to be easier.

This puzzle deals with mappings between the sixteen positions and the sixteen labels on the blocks (treating the non-block as 0). The legal moves require information from both the set of positions and the set of labels, since the zero block must move, and it can only be interchanged with a block whose position is adjacent in the array. To keep track of the location of 0, we introduce sixteen different objects which are four by four arrays with one position marked. The legal moves are then mappings between these sets, and they can be described entirely in terms of blocks of one object are taken to blocks of the other. The mappings then corresponds exactly to the puzzle moves, and they can be composed only if the first move leaves the puzzle in a state that allows the second move. The action of playing with the puzzle consists of composing these basic mappings, and the effect of these moves is recorded in the appearance of the puzzle after making the moves. If the puzzle is set aside, and then a new sequence of moves made, the total effect will be the composition of the mappings described by the two sequences of moves. Such a collection of mappings that can only be composed if they involve a common object, but in which every mapping has an inverse, is called a *groupoid*.

**(a)** How many basic mappings (i.e., puzzle moves) are there in the 15-puzzle groupoid?

Further mappings are formed by composing several basic mappings, but it is best to create these mappings in a systematic fashion. In particular, a group can be obtained by choosing a base object and considering those mappings that start and end with the blank cell in the position that defines this object. Then, a designated path of mappings connecting each other object to the base (assuming that there is *some* way to get between the base object and the selected one). For the 15-puzzle, one can take the base object to have the blank in the lower right corner, and connect other objects to it by moves that go up the last column until they get to the correct row, and then go across the row. The designated paths give mappings that show that the groups of composed mappings from any object to itself are all isomorphic to one another (again, assuming that there is a path connecting each object to the base object).

Given a mapping in the groupoid, a mapping from the base object to itself is obtained by composing the designated path to the domain of mapping, the mapping itself, and the inverse of the designated path to the codomain. This construction can be made for any element of the groupoid.

**(b)** For the 15-puzzle groupoid, how many of the mappings determined by applying this construction to the basic moves reduce exactly to the identity?

The mappings of the base object to itself can be described entirely in terms of moving a block from one location to another (that is, as an  $A(S)$ ) and do not need to refer to a labelling of cells.

**(c)** Describe the permutation of blocks on our selected base object that results from moving a block in the first position of the second row into a space in the upper left corner.

**(d)** The importance of a study of the basic moves is due to this construction being a homomorphism for this structure. In particular, show that, if we have two composable elements of the groupoid, the image of their composition is the composition of their images in the group of permutations of the base object.

This can be used to characterize the allowable permutations of the base object. It will turn out that this group is always a subgroup of index 2 in the  $S_{15}$  of all permutations of the 15 blocks in this object.