Conjugacy in groups

Let G be a group. Two elements a, b in G are said to be *conjugate* if there is an element $x \in G$ satisfying

$$xax^{-1} = b$$

Conjugacy is an *equivalence relation* and the equivalence classes are called *conjugacy classes.*

- In a commutative group, conjugate elements are equal and all of the conjugacy classes have one elements.
- In the group of invertible matrices $\mathrm{GL}(n, \mathbb{R})$, conjugacy corresponds to a change of basis:

$$XAX^{-1} = B$$

For example if B has a basis of eigenvectors, and X is a matrix whose columns form such a basis, and A is the diagonal matrix whose entries are the corresponding eigenvalues, then we have $BX = XA$ (the columns on both sides are the columns of X multiplied by the corresponding eigenvalues) and so $XAX^{-1} = B$.

- In the group S_n, two permutations are conjugate if they have the same *shape.* This is what we explore here.

The *shape* of a permutation is the set of all its cycle lengths (including cycles of length 1, or fixed points). For example, the shape of the permutation (1357)(28) considered as a permutation in S_{10} is $\{1, 1, 1, 1, 2, 4\}$. These numbers add up to $n = 10$.

The possible shapes of a permutation in S_n are the number of ways of expressing n as a sum of positive integers. For example, in S_6 we have the following 11 shapes:

$$\begin{gathered}6;\ 5+1;\ \ 4+2;\ \ 4+1+1;\ \ 3+3;\ \ 3+2+1;\ \ 3+1+1+1;\ \ 2+2+2;\\ 2+2+1+1;\ \ 2+1+1+1+1+1;\ \ 1+1+1+1+1+1\end{gathered}$$

The number of ways to write n as a sum of positive integers is called $p(n)$, the *partition function.* So for example $p(6) = 11$.

Typical elements of these shapes are:

$$(123456);\ (12345);\ (1234)(56);\ (123)(456);\ (123)(45);\ (123);\ (12)(34)(56);\ (12);\ e$$

The order of a cycle is its length. The order of a disjoint product of cycles is the least common multiple of their lengths. The orders of the 11 elements shown are therefore:

$$6;\quad 5;\quad 4;\quad 3;\quad 6;\quad 3;\quad 2;\quad 2;\quad 1$$

and the same applies to any conjugate element.

When conjugating a permutation σ by a permutation τ, one applies τ to the cycle representation of σ. Thus for example if $\tau = (123456)$ in S_6, one replaces each element by the next (with 6 replaced by 1). So if $\sigma_1 = (135)$ and $\sigma_2 = (153)$ then

$$\tau\sigma_1\tau^{-1} = (246); \tau\sigma_2\tau^{-1} = (264)$$

Similarly, given σ_1 and σ_2 of the same shape, one finds a solution τ to the conjugacy equation

$$\tau\sigma_1\tau^{-1} = \tau\sigma_2\tau^{-1}$$

by inspection. For example, if $\sigma_1 = (12345)$ and $\sigma_2 = (13524)$ then we can take $1 \to 1$ and we are forced after that to take $2 \to 3$, $3 \to 5$, $4 \to 2$, $5 \to 4$ or in other words $\tau = (2354)$. There are other possibilities: since $(13524) = (35241)$ we could take for example $\tau = (1325)$. All together there are five possible choices for τ.

What about conjugacy classes in A_6? A cycle of length k can be written as a product of $k-1$ transpositions, so cycles of even length are odd permutations, and cycles of odd length are even permutations. So: a shape corresponds to a conjugacy class in A_6 if and only if the number of even numbers in the shape is even: 0 or 2:

$$5; 4+2; 3+3; 3+1+1+1; 2+2+1+1+1+1; 1+1+1+1+1+1$$

However these are conjugacy classes of S_6 contained in A_6, not necessarily conjugacy classes for the group A_6. But with one exception they turn out to be conjugacy classes for the group A_6.

Theorem

Let σ be an element of A_n and suppose σ commutes with an odd permutation in S_n. Then the conjugacy class of σ in A_n coincides with its conjugacy class in S_n.

Proof: Let τ be an odd permutation commuting with σ. Then for any element σ' conjugate to σ by an element $x \in S_6$, we have

$$(x\tau)\sigma(x\tau)^{-1} = x\sigma x^{-1} = \sigma'$$

Now since exactly one of the permutations x and $x\tau$ will be an even permutation, this shows that σ and σ' are also conjugate in A_n.

In particular any permutation that involves a cycle of even length satisfies this condition: for example $(12)(3456)$ in A_6 commutes with both (12) and (3456) and each of these is an odd permutation. Any permutation that contains two cycles of equal and odd length also satisfies our condition: for example $(123) = (123)(4)(5)$ (two cycles of length 1) commutes with (45), and $(123)(456)$ commutes with $(14)(25)(36)$. This leaves over the case of permutations whose cycles all have odd length, and with all lengths different (in particular, at most one fixed point). In A_6 this means 5-cycles.

In fact, the 5-cycles in A_6 fall into two conjugacy classes, represented by (12345) and (12354). To see this, one shows first that no odd permutation commutes with (12345) and then that this prevents (12345) and (12354) being conjugate.

In Hungerford §8.4 one finds the following result: If G is a group and $a \in G$ an element, then the number of elements conjugate to a is the quotient

$$|G|/|C(a)|$$

where $C(a)$ is the *centralizer* of a in G, defined as

$$\{g \in G : ga = ag\}$$

To explore this further, one may compute the number of elements of S_7 conjugate to $(12)(34)(56)$: the formula given leads to the result

$$\frac{7!}{2^3 \cdot 3!} = 105$$