## The construction of a regular pentagon

We treat only the *algebraic* part of the problem, not the question of finding a good geometrical construction. Euclid solves the problem geometrically in Book IV of *The Elements of Geometry* (written in Alexandria, Egypt, in Greek, about 300 B.C., based on material that had probably been known for several centuries before that).

Algebraically, we need to find a solution in $\mathbb{C}$ to the conditions

$$z^5 = 1; \qquad z \neq 1$$

This is a root of the *cyclotomic* polynomial $\dfrac{z^5-1}{z-1} = z^4+z^3+z^2+z+1$. Recall that this polynomial is irreducible, so that the degree $[\mathbb{Q}[z]:\mathbb{Q}] = 4$ for any root of this polynomial. Since 4 is a power of 2, we cannot say definitely whether this number is constructible, but we may *suspect* that it is. (Actually, most irreducible equations of degree 4 do not have constructible roots, but this one is very special.)

First, an exercise about quadratic equations.

*Exercise 1*

*Suppose* $r+s=a$ *and* $rs=b$. *Then* $r$ *and* $s$ *are the roots of the quadratic equation* $x^2-ax+b=0$.

Now, here is Gauss' method for solving the equation using square roots.

(1) Work out the subgroups of $U_5$, and place them one inside another:

$$\{1\}; \{1,4\}; \{1,2,3,4\}$$

(2) Arrange the elements of $U_5$ so that the subgroups can be seen, going from left to right:

$$[1|4|2,3]$$

Consider the following **numbers** associated with these **groups**: $a = z+z^4+z^2+z^4$, $b_1 = z+z^4$, $b_2 = z^2+z^3$, $c_{11} = z$, $c_{12} = z^4$, $c_{21} = z^2$, $c_{22} = z^3$.

Let's look at what we have done. In each case, we sum certain powers of $z$, and there is a rule for which powers we take.

$a$: This corresponds to the subgroup $\{1,4,2,3\} = U_5$ and we take all powers, accordingly.

$b$: Here $b_1$ corresponds to the subgroup $\{1,4\}$ and $b_2$ corresponds to the "other" elements $\{2,3\}$.

$c$: Here $c_{11}$ corresponds to the tiny subgroup $\{1\}$ so $c_{11}$ is just $z$. $c_{12}$ is the "other" element of $\{1,4\}$.

Similarly, $c_{22}, c_{23}$ breaks up $b_2$ the way $c_{11}, c_{12}$ breaks up $b_1$.

What is so special about these numbers?

**First**, $a$ is rational, in fact $a = -1$.

**Second**, $b_1$ and $b_2$ are roots of a single quadratic equations with rational coefficients. To see this, use exercise 1: it is enough to check that $b_1+b_2$ and $b_1 \cdot b_2$ are rational.

$b_1+b_2 = a$ (this is the point), and we know that is rational.

$b_1 \cdot b_2 = (z+z^4)\cdot(z^2+z^3) = z^3+z^4+z+z^2 = a$ as well (slightly miraculous, and we need to analyze this).

**Third**, $c_{11}$ and $c_{12}$ satisfy a quadratic equation with coefficients in $\mathbb{Q}[b_1,b_2]$, and so do $c_{21}$ and $c_{22}$.

Let's check the sums and products.

Sums: $c_{11} + c_{12} = b_1$, $c_{21} + c_{22} = b_2$. That is easy enough, as we set it up that way.
Products $c_{11} \cdot c_{12} = 1$, $c_{21} \cdot c_{22} = 1$, slightly miraculous again.

Anyway, since $c_{11} = z$, we know now that $z$ is constructible from $b_1$ and $b_2$, and $b_1, b_2$ are constructible, so $z$ is constructible.

Let's work out the formulas.

$b_1 + b_2 = -1$, $b_1 b_2 = -1$, so these are roots of the quadratic equation

$$x^2 + x - 1 = 0$$

that is

$$\frac{-1 \pm \sqrt{5}}{2}$$

Secondly $c_{11} + c_{12} = b_1$, $c_{11} \cdot c_{12} = 1$, so these are roots of the quadratic equation

$$x^2 - b_1 x + 1 = 0$$

that is

$$\frac{b_1 \pm \sqrt{b_1^2 - 4}}{2}$$

All together we get 4 possible roots, which I calculate as follows (this should be checked):

$$\frac{-1 \pm \sqrt{5} \pm i\sqrt{10 \pm 2\sqrt{5}}}{4}$$

In particular one finds

$$\cos(72^\circ) = \frac{-1 + \sqrt{5}}{4}; \qquad \sin(72^\circ) = \sqrt{\frac{10 + 2\sqrt{5}}{4}}$$

assuming we have calculated properly up to this point.