Mathematics 351. Review sheet II

1 Determine the following dimensions (rigorously), and exhibit a basis in each case.
(*a*) $[\mathbb{Q}[\sqrt[3]{15}] : \mathbb{Q}]$.
(*b*) $[\mathbb{Q}[\sqrt[3]{25}] : \mathbb{Q}]$.
(*c*) $[\mathbb{Q}[\sqrt[3]{125}] : \mathbb{Q}]$.
(*d*) $[\mathbb{Q}[\sqrt{2}, \sqrt{3}, i] : \mathbb{Q}]$.

2
(*a*) Let $\sigma = (1234)$, $\tau = (5678)$ in S_8. Determine the order of the three elements σ, τ, $\sigma\tau$, as well as the order of their inverses and their squares.
Determine the order of the group $\langle \sigma, \tau \rangle$ generated by σ and τ.
Does σ belong to the cyclic subgroup generated by $\sigma\tau$?
(*b*) Repeat with $\sigma = (1234)$, $\tau = (567)$ in S_7.

3 Working in the group U_{19}, find a generator for the group and find all elements of order 3 in the group.

4 Determine the order of each of the following groups, and find an element of order 5 if there is one:

$$U_{11},\ U_{15},\ U_{16},\ U(\mathbb{Z}),\ S_6,\ \mathrm{GL}(2, \mathbb{Z}_3)$$

For $\mathrm{GL}(2, \mathbb{Z}_3)$: recall that an invertible matrix has linearly independent columns. So the first column can be any nonzero vector and the second column should be linearly independent from it. (This leads to the answer 48 in this case.)

5 Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ in $\mathrm{GL}(2, \mathbb{Z}_{11})$. Find the order of A.

6 Find the length of the period in the decimal expansion of $1/13, 1/17, 1/19$. And also for the binary expansion.

7 Factor 210 into irreducible factors in $\mathbb{Z}[i]$.

8 Show that there are no points $P, Q \in \mathbb{Q}^2$ whose distance is $\sqrt{3}$.

9 Find the g.c.d. of $1 + 3i$ and $5 + 3i$ in $\mathbb{Z}[i]$, using the Euclidean algorithm.

10 Let $f : R \to S$ be a homomorphism, and J an ideal of S. Show that the inverse image $f^{-1}(J)$ is an ideal in R.
(Definition: $f^{-1}(J) = \{r \in R : f(r) \in J\}$.)

11 Let R, S be rings and let $I \subseteq R$, $J \subseteq S$ be ideals of those rings. Show that $I \times J$ is an ideal in $R \times S$.

12 Let G be a group in which the law

$$(ab)^{-1} = a^{-1}b^{-1}$$

holds. Show that G is commutative.

13 Let H be a subgroup of G. Prove that $e_H = e_G$.

14 Let G be a group of even order. Prove that G contains an element of order 2.

15 Let a, b be elements of a group G. Show that ab and ba have the same order.

16 Let p be a prime, n an integer, and suppose $4n^2 + 1$ is divisible by p. Show that $p \equiv 1 \mod 4$.

17 Let $u \in \mathbb{C}$ and let I_u be the set of polynomials p with rational coefficients for which $p(u) = 0$.
(*a*) Show that I_u is an ideal in the ring $\mathbb{Q}[x]$.
(*b*) Determine this ideal in the following cases: $u = 0$; $u = i$; $u = \sqrt{2}$; $u = \pi$.
(*c*) Prove that $\mathbb{Q}[x]/I_u \simeq \mathbb{Q}[u]$.