Mathematics 351. Workshop #10

1 Act Normal

Let G be a group and K and N two normal subgroups. Show that $K \cap N$ and KN are normal subgroups of G.

2 Cubes and squares

(a) Show that any cubic equation of the form $x^3 + ax + b = 0$ has the solution $x = \sqrt[3]{\frac{-b}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{-b}{2} - \sqrt{\Delta}}$, where $\Delta = \frac{4a^3+27b^2}{108}$. (Just substitute $x = \sqrt[3]{u} + \sqrt[3]{v}$, and see if you can find a quadratic equation for u and v.)

(b) Use this formula to solve the cubic equations $x^3+x-2 = 0$ (roots 1 and $\frac{1 \pm i\sqrt{7}}{2}$), and $x^3-7x+6 = 0$ (roots $1, 2, -3$). Both are surprisingly difficult, even with the formula.

3 $x^2 = 1$

Suppose that G is a group satisfying the law $x^2 = 1$.

(a) Show that G can be considered as a vector space over the field $\mathbb{Z}_2$.

(b) Show that $G \simeq \mathbb{Z}_2^d$ for some d.

(c) Show that a group of order 4 is isomorphic to one of the following: $\mathbb{Z}_2 \times \mathbb{Z}_2$; $\mathbb{Z}_4$.

(*Hint:* either every element has order 2, or some element has order 4.)

(d) Show that any group G of order 6 is isomorphic either to $\mathbb{Z}_6$ or to S_3.

Analysis: there is an element a of order 2 since $|G|$ is even, and an element b of order 3 (otherwise part (b) applies). Show next that aba^{-1} is either b or b^{-1}. In the first case show $G \simeq \mathbb{Z}_6$ and in the second case show $G \simeq S_3$.

4 $S_6 \neq S_6$

Construction of an unusual automorphism of S_6.

Outline:

Let $A = \{1, 2, 3, 4, 5, 6\}$. Think of the elements of A as the vertices of regular hexagon. Let E be the set of all possible line segments between two of these vertices; E has 15 elements and S_6 acts naturally on E. The elements of E are called edges.

Now let a 2-factor be a triple of edges such that each vertex in A lies on exactly one edge. There are 15 2-factors (there are examples below). Let F be the set of 2-factors.

A factorization is a set of five 2-factors such that every edge belongs to exactly one of the 2-factors. There are 6 factorizations. Let A^* be the set of factorizations.

S_6 acts naturally on E, F, and A^*. Since A^* has 6 elements, if we label them $1, 2, 3, 4, 5, 6$, then this gives a homomorphism from S_6 back into S_6.

Details:

(a) Find all the factorizations. Here is one:

$$[\{1,2\},\{3,4\},\{5,6\}] \quad [\{1,3\},\{2,5\},\{4,6\}] \quad [\{1,4\},\{2,6\},\{3,5\}]$$
$$[\{1,5\},\{2,4\},\{3,6\}] \quad [\{1,6\},\{2,3\},\{4,5\}]$$

Now operate on this by the cycle $(12)(345)$ five times to find the other 5.

(b) Now label the six factorizations $1, 2, 3, 4, 5, 6$ and then work out how (12) and (123) operate on this set of six elements. Notice that in the new action, each permutation "acts like" a permutation with a completely different structure.