Supplementary notes on duality

Math 354, Spring 2006

This is a supplement to Sections 3.1, 3.2, and 3.3. It addresses the Duality Theorem of Section 3.2 from the concrete perspective of tableaux. It gives more details about the important idea of complementary slackness.

A later supplement will introduce the "primal-dual" method, a popular variant of the simplex method, not covered in the text.

Let's say that we have a LPP (P) in standard form, and its dual (D):

(P)	(D)
Maximize $z_P = \boldsymbol{c}^T\boldsymbol{x}$	Minimize $z_D = \boldsymbol{b}^T\boldsymbol{w}$
subject to	subject to
$A\boldsymbol{x} \leq \boldsymbol{b}$	$A^T\boldsymbol{w} \geq \boldsymbol{c}$
$\boldsymbol{x} \geq \boldsymbol{0}$	$\boldsymbol{w} \geq \boldsymbol{0}$

Here A is any $m \times n$ matrix, $\boldsymbol{x} = [x_1 \cdots x_n]^T$ is the column vector of primal variables and $\boldsymbol{w} = [w_1 \cdots w_m]^T$ is the column vector of dual variables. Also $\boldsymbol{c}$ is a column vector in $\boldsymbol{R}^n$ and $\boldsymbol{b}$ a column vector in $\boldsymbol{R}^m$, each playing the role of "cost vector" in one problem and "resource vector" in the other.

1. Standard Form, Slack Variables, Canonical Form

In this totally pedestrian section we prepare for further analysis by putting (D) in standard form, and then putting both problems in canonical form. Nothing new is happening here, but keep your eye on the notation. The reason is that slack variables will turn out to be a key connector between (P) and (D), so we will use carefully chosen notation for them that is a bit unorthodox.

First, putting (D) in standard form means changing the minimization problem to a maximization problem, and reversing the inequality sign in the constraint $A\boldsymbol{w} \geq \boldsymbol{c}$. Our problems then read as follows. The resulting problem (D') is equivalent to (D).

(P)	(D')
Maximize $z_P = \boldsymbol{c}^T\boldsymbol{x}$	Maximize $-z_D = (-\boldsymbol{b})^T\boldsymbol{w}$
subject to	subject to
$A\boldsymbol{x} \leq \boldsymbol{b}$	$(-A^T)\boldsymbol{w} \leq (-\boldsymbol{c})$
$\boldsymbol{x} \geq \boldsymbol{0}$	$\boldsymbol{w} \geq \boldsymbol{0}$

Next, canonical form. For (P), introduce slack variables $w_1^*, \ldots, w_m^*$, for the constraints numbered $1, \ldots, m$, respectively. (The names are an important aid in keeping track of the situation.) Likewise introduce slack variables for (D'), called $x_1^*, \ldots, x_n^*$. In canonical form, (P) and (D') are

(P)	(D')
Maximize $z_P = \boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{C}^T\boldsymbol{X}$	Maximize $-z_D = -\boldsymbol{b}^T\boldsymbol{w} = -\boldsymbol{B}^T\boldsymbol{W}$
subject to	subject to
$[A \mid I]\boldsymbol{X} = \boldsymbol{b}$	$[I \mid -A^T]\boldsymbol{W} = -\boldsymbol{c}$
$\boldsymbol{X} \geq \boldsymbol{0}$	$\boldsymbol{W} \geq \boldsymbol{0}$

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ w_1^* \\ \vdots \\ w_m^* \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_1 \\ \vdots \\ b_m \end{bmatrix}, \quad \boldsymbol{W} = \begin{bmatrix} x_1^* \\ \vdots \\ x_n^* \\ w_1 \\ \vdots \\ w_m \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The unorthodox thing is that **we have put the slack variables for (D') first in $\boldsymbol{W}$, instead of last**. This is very helpful for this theoretical discussion.

2. Duality Gap; the Weak Duality Theorem

Suppose that $\boldsymbol{X}$ and $\boldsymbol{W}$ as above are solutions to the canonical form versions of (P) and (D'). Then the slack variables $w_1^*, \ldots, w_m^*$ make up the difference between $A\boldsymbol{x}$ and $\boldsymbol{b}$. Therefore

$$\begin{bmatrix} w_1^* \\ \vdots \\ w_m^* \end{bmatrix} = \boldsymbol{b} - A\boldsymbol{x}, \text{ and so } \boldsymbol{X} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{b} - A\boldsymbol{x} \end{bmatrix}.$$

Similarly $x_1^*, \ldots, x_n^*$ make up the difference between $\boldsymbol{c}$ and $A^T\boldsymbol{w}$. Therefore

$$\begin{bmatrix} x_1^* \\ \vdots \\ x_n^* \end{bmatrix} = A^T\boldsymbol{w} - \boldsymbol{c}.$$

This implies the useful "Duality Gap" identity:

$$\text{(A)} \qquad \begin{aligned} \boldsymbol{X} \bullet \boldsymbol{W} &= (-\boldsymbol{c} + A^T\boldsymbol{w})^T\boldsymbol{x} + \boldsymbol{w}^T(\boldsymbol{b} - A\boldsymbol{x}) = -\boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{w}^T\boldsymbol{b} \\ \boldsymbol{X} \bullet \boldsymbol{W} &= -\boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{b}^T\boldsymbol{w}. \end{aligned}$$

Furthermore, $\boldsymbol{x}$ is a feasible solution of (P) if and only if $\boldsymbol{X} \geq \boldsymbol{0}$, and $\boldsymbol{w}$ is a feasible solution of (D) if and only if $\boldsymbol{W} \geq \boldsymbol{0}$. An immediate consequence of these observations is the Weak Duality Theorem, which in turn has some important corollaries.

Weak Duality Theorem. *Let $\boldsymbol{x}$ and $\boldsymbol{w}$ be* **feasible** *solutions of (P) and (D), respectively. Then*

$$z_D(\boldsymbol{w}) \geq z_P(\boldsymbol{x}), \text{ that is, } \boldsymbol{b}^T\boldsymbol{w} \geq \boldsymbol{c}^T\boldsymbol{x}.$$

Proof. Since $\boldsymbol{x}$ and $\boldsymbol{w}$ are feasible, $\boldsymbol{X} \geq \boldsymbol{0}$ and $\boldsymbol{W} \geq \boldsymbol{0}$. Hence, $\boldsymbol{X} \bullet \boldsymbol{W} \geq 0$. By the Duality Gap, $\boldsymbol{b}^T\boldsymbol{w} \geq \boldsymbol{c}^T\boldsymbol{x}$.

Corollary 1. *If either* (P) *or* (D) *is an unbounded LPP, then the other one is an infeasible LPP.*

Corollary 2. *Suppose that* $\boldsymbol{x}_0$ *is a feasible solution of* (P)*, and that* $\boldsymbol{w}_0$ *is a feasible solution of* (D)*. Suppose also that* $\boldsymbol{b}^T\boldsymbol{w}_0 = \boldsymbol{c}^T\boldsymbol{x}_0$*. Then* $\boldsymbol{x}_0$ *and* $\boldsymbol{w}_0$ *are optimal solutions of* (P) *and* (D)*, respectively.*

Proof. Let's show that $\boldsymbol{x}_0$ is an optimal solution for (P). We know it's a feasible solution. Let $\boldsymbol{x}$ be any feasible solution for (P). Then

$$\boldsymbol{c}^T\boldsymbol{x}_0 = \boldsymbol{b}^T\boldsymbol{w}_0 \geq \boldsymbol{c}^T\boldsymbol{x},$$

by assumption $(=)$ and the Weak Duality Theorem $(\geq)$. Since this holds for any feasible solution $\boldsymbol{x}$, $\boldsymbol{x}_0$ is an optimal solution of (P).

The proof for $\boldsymbol{w}_0$ is similar.

3. Complementary Slackness

Again given a pair of vectors $\boldsymbol{X}$ and $\boldsymbol{W}$ as above, they may or may not have an important property known as **complementary slackness**, or just **complementarity**. This property stipulates that

$$x_i x_i^* = 0 \text{ and } w_j^* w_j = 0 \text{ for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, m.$$

In other words, this property holds if and only if for each of the $n + m$ positions, one or both of the vectors $\boldsymbol{X}$ and $\boldsymbol{W}$ has a 0 entry. Each position corresponds to a variable x_i or w_j for (P) or (D), and the slack variable x_i^* or w_j^* for the constraint in the dual problem.

So complementary slackness means that if some variable for (the standard form problem) (P) or (D) is nonzero, then both sides of the corresponding constraint inequality in (D) or (P) are equal, i.e., the corresponding slack variable for (D) or (P) equals 0.

Complementarity clearly implies that $\boldsymbol{X} \bullet \boldsymbol{W} = 0$. On the other hand if $\boldsymbol{X} \bullet \boldsymbol{W} = 0$, then complementarity need not hold, because $\boldsymbol{X}$ and $\boldsymbol{W}$ can have negative entries. **However**, if $\boldsymbol{X} \geq 0$ and $\boldsymbol{W} \geq 0$ and $\boldsymbol{X} \bullet \boldsymbol{W} = 0$, then $\boldsymbol{X}$ and $\boldsymbol{W}$ are complementary. In this case the dot product is a sum of nonnegative terms, and so can only be zero if all the terms are zero.

Complementarity and optimality are closely related.

Complementary Slackness Theorem. *Let $\boldsymbol{x}$, $\boldsymbol{w}$, $\boldsymbol{X}$ and $\boldsymbol{W}$ be as above. Suppose that $\boldsymbol{x}$ and $\boldsymbol{w}$ are* **feasible** *solutions for (P) and (D), respectively. Then*

$$\boldsymbol{x} \text{ and } \boldsymbol{w} \text{ are optimal solutions for } (P) \text{ and } (D) \text{ (resp.)}$$
$$\Longleftrightarrow$$
$$\boldsymbol{X} \text{ and } \boldsymbol{W} \text{ satisfy complementary slackness.}$$

Furthermore, when this occurs, $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{w}$.

Proof of $\Longleftarrow$. Complementary slackness implies $\boldsymbol{X} \bullet \boldsymbol{W} = 0$. By the Duality Gap $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{w}$. By Corollary 2 to the Weak Duality Theorem, $\boldsymbol{x}$ and $\boldsymbol{w}$ are optimal solutions of (P) and (D), respectively.

The proof of $\Longrightarrow$ is harder and will be given below.

4. Corresponding Basic Solutions of (P) and (D)

A basic solution X to (P) has m basic variables and n non-basic variables; a basic solution W to (D) has n basic variables and m nonbasic variables. Since nonbasic variables are 0, X and W could be a perfect fit for complementary slackness, with X having basic variables just where W has nonbasic variables (and vice-versa).

Definition. *Let X and W be basic solutions to (P) and (D), respectively. We will say that X and W* **correspond** *if and only if for each $i = 1, \ldots, n$ and each $j = 1, \ldots, m$, exactly one of x_i and x_i^* is a basic variable and exactly one of w_j and w_j^* is a basic variable.*

Corresponding basic solutions are **very** closely related, in the following ways.

PROPERTY 1. Corresponding basic solutions $\boldsymbol{X}$ and $\boldsymbol{W}$ of (P) and (D) satisfy complementary slackness.

Property (1) is not hard to see, because for each i, either x_i or x_i^* is nonbasic, so either $x_i = 0$ or $x_i^* = 0$, so $x_i x_i^* = 0$.

PROPERTY 2. For each basic solution $\boldsymbol{X}$ of (P) (and choice of basic variables), there is a unique corresponding basic solution $\boldsymbol{W}$ of (D) (and choice of basic variables).

Indeed the basic variables of $\boldsymbol{X}$ being given, the basic variables of $\boldsymbol{W}$ are then determined, and so $\boldsymbol{W}$ is determined.

PROPERTY 3. Let $\boldsymbol{X}$ and $\boldsymbol{W}$ be corresponding basic solutions of (P) and (D). Let $T_{\boldsymbol{X}}$ and $T_{\boldsymbol{W}}$ be tableaux for (P) and (D) corresponding to $\boldsymbol{X}$ and $\boldsymbol{W}$, respectively. Delete (or ignore) the basic columns of $T_{\boldsymbol{X}}$ and $T_{\boldsymbol{W}}$ to obtain arrays $T^o_{\boldsymbol{X}}$ and $T^o_{\boldsymbol{W}}$. Then

(a) The values of z_0 in $T^o_{\boldsymbol{X}}$ and $T^o_{\boldsymbol{W}}$ are negatives of each other.

(b) The resource column of $T^o_{\boldsymbol{X}}$ is "identical" to the transpose of the objective row of $T^o_{\boldsymbol{W}}$, and vice-versa.

(c) The central portion of $T^o_{\boldsymbol{X}}$ is "identical" to the negative transpose of the central portion of $T^o_{\boldsymbol{W}}$.

The word "identical" is in quotation marks because (b) and (c) are not literally true. Let's use the following convention: if g is one of the variable names for $\boldsymbol{X}$, then g' is the corresponding variable name for $\boldsymbol{W}$ – obtained either by adding or removing the asterisk. So if $g = x_i$ then $g' = x_i^*$; if $g = w_j^*$ then $g' = w_j$. In (c), every entry a_{gh} of $T^o_{\boldsymbol{X}}$ is in a row labelled by a variable g in $\boldsymbol{X}$, and a column labelled by a variable h in $\boldsymbol{X}$. A similar labelling system holds for entries $a'_{g'h'}$ in $T^o_{\boldsymbol{W}}$. What (c) asserts is that

$$a_{gh} = -a'_{h'g'}, \text{ for all basic } g \text{ and nonbasic } h \text{ in } \boldsymbol{X}.$$

Notice that g' is nonbasic in $\boldsymbol{W}$ and h' is basic in $\boldsymbol{W}$.

Similarly (b) means that the objective row entry of $T^o_{\boldsymbol{X}}$ for a nonbasic variable g equals the resource column entry of $T^o_{\boldsymbol{W}}$ for the (basic!) variable g'. And a similar connection holds between the objective row of $T^o_{\boldsymbol{W}}$ and the resource column of $T^o_{\boldsymbol{X}}$.

Properties (2) and (3) are not obvious. Their proofs are given in exercises below.

Here is an example. It is only to illustrate the above properties; we are not actually going to solve a LPP. Consider the following primal-dual pair:

(P)	(D')
Maximize $2x_1 - 3x_2$	Maximize $-3w_1 + 2w_2 + w_3$
subject to	subject to
$4x_1 - x_2 \leq 3$	$-4w_1 - 3w_2 + w_3 \leq -2$
$3x_1 + 5x_2 \leq -2$	$w_1 - 5w_2 - 2w_3 \leq 3$
$-x_1 + 2x_2 \leq -1$	$w_1 \geq 0$
$x_1 \geq 0$	$w_2 \geq 0$
$x_2 \geq 0$	$w_3 \geq 0$

After introducing the slack variables, we can consider the initial tableaux:

(P)

	x_1	x_2	w_1^*	w_2^*	w_3^*	
w_1^*	4	−1	1	0	0	3
w_2^*	3	5	0	1	0	−2
w_3^*	−1	2	0	0	1	−1
	−2	3	0	0	0	0

(D')

	x_1^*	x_2^*	w_1	w_2	w_3	
x_1^*	1	0	−4	−3	1	−2
x_2^*	0	1	1	−5	−2	3
	0	0	3	−2	−1	0

Notice that if you ignore the basic columns and ignore stars, the tableaux are transposes of each other, with minus signs inserted in the core and the objective function value z_0. This is what (3) says.

Now let's consider another basic solution for (P), say the one with basic variables x_1, x_2, w_3^*. The corresponding basic solution for (D') has basic variables w_1, w_2. Using row operations to create a 3×3 identity matrix in the x_1, x_2, w_3^* columns of the initial (P) tableau, we get the (P) tableau for the new basic solution. Similarly, the new (D') tableau comes from pivoting in the initial (D') tableau, to make a 2×2 identity matrix in the w_1, w_2 columns. The new pair of corresponding tableaux is

(P)

	x_1	x_2	w_1^*	w_2^*	w_3^*	
x_1	1	0	$\frac{5}{23}$	$\frac{1}{23}$	0	$\frac{13}{23}$
x_2	0	1	$-\frac{3}{23}$	$\frac{4}{23}$	0	$-\frac{17}{23}$
w_3^*	0	0	$\frac{11}{23}$	$-\frac{7}{23}$	1	$\frac{24}{23}$
	0	0	$\frac{19}{23}$	$-\frac{10}{23}$	0	$\frac{77}{23}$

(D')

	x_1^*	x_2^*	w_1	w_2	w_3	
w_1	$-\frac{5}{23}$	$\frac{3}{23}$	1	0	$-\frac{11}{23}$	$\frac{19}{23}$
w_2	$-\frac{1}{23}$	$-\frac{4}{23}$	0	1	$\frac{7}{23}$	$-\frac{10}{23}$
	$\frac{13}{23}$	$-\frac{17}{23}$	0	0	$\frac{24}{23}$	$-\frac{77}{23}$

and again (3) is true–look at the numbers!

5. Feasibility and the Optimality Criterion

We can draw several useful conclusions from these relationships.

We know that each tableau for a LPP corresponds to a basic solution of that LPP. Also, a basic solution is feasible if and only if the corresponding tableau is "feasible" in the sense that all entries of the resource column are ≥ 0 (except possibly z_0).

We'll call a tableau "dual-feasible" if and only if all entries of the objective row are ≥ 0 (except possibly z_0).

Property (3) makes this reasonable. As before, let $T_{\boldsymbol{X}}$ and $T_{\boldsymbol{W}}$ be tableaux for corresponding basic solutions of (P) and (D'). Then by Property (3), $T_{\boldsymbol{X}}$ is dual-feasible if and only if $T_{\boldsymbol{W}}$ is feasible, and vice-versa.

We used to call dual-feasibility the "optimality criterion". This was when we were dealing only with **feasible** tableaux. It is the tableaux that are **both feasible and dual-feasible** that correspond to an optimal solution of the underlying LPP. Non-feasible tableaux are "out of bounds" for the optimality criterion. The proper statement of the optimality criterion is: If a **feasible** tableau satisfies the optimality criterion, then it corresponds to an optimal solution. The common terminology is misleading.

We can add three more properties to our list.

PROPERTY 4. For the pair $T_{\boldsymbol{X}}$ and $T_{\boldsymbol{W}}$, one of them is feasible if and only if the other one is dual-feasible.

PROPERTY 5. $T_{\boldsymbol{X}}$ is both feasible and dual-feasible if and only if $T_{\boldsymbol{W}}$ is both feasible and dual-feasible. In this case, the tableaux represent optimal solutions of (P) and (D'), and the optimal values of (P) and (D) are equal.

(In Property (5), the z_0 entries of $T_{\boldsymbol{X}}$ and $T_{\boldsymbol{W}}$ are negatives of each other, but remember that to convert (D) to (D'), we changed a minimization problem changed to a maximization by negating the objective function. So the optimal values of (P) and (D) are equal.)

PROPERTY 6. If either (P) or (D) has an optimal solution, then so does the other, and the optimal values of the two objective functions coincide.

(In Property (6), if (P) has an optimal solution, then the two-phase method will find an optimal solution. Therefore there's a feasible and dual-feasible tableau $T_{\boldsymbol{X}}$. With the corresponding tableau $T_{\boldsymbol{W}}$ for (D'), Property (5) applies and tells us what we want.)

Now we can prove the $\Longrightarrow$ direction of the Complementary Slackness Theorem: if $\boldsymbol{x}$ and $\boldsymbol{w}$ are optimal solutions to (P) and (D') as above, then $\boldsymbol{X}$ and $\boldsymbol{W}$ are complementary.

By assumption $\boldsymbol{x}$ and $\boldsymbol{w}$ are certainly feasible, so $\boldsymbol{X} \geq \boldsymbol{0}$ and $\boldsymbol{W} \geq \boldsymbol{0}$. Indeed $\boldsymbol{x}$ and $\boldsymbol{w}$ are optimal so by Property (6), $\boldsymbol{b}^T\boldsymbol{w} = \boldsymbol{c}^T\boldsymbol{x}$. Therefore by the Duality Gap, $\boldsymbol{X} \bullet \boldsymbol{W} = 0$. But $\boldsymbol{X} \geq \boldsymbol{0}$ and $\boldsymbol{W} \geq \boldsymbol{0}$, so $\boldsymbol{X}$ and $\boldsymbol{W}$ must be complementary. (The dot product is a sum of nonnegative terms, and is 0, so each term must be 0.)

6. The Duality Theorem

The Duality Theorem is the most important theorem in linear optimization. It comes from Property (6), Corollary 1 to the Weak Duality Theorem, and part of the Complementary Slackness Theorem. You should convince yourself that with these results, the Duality Theorem is completely proved.

Duality Theorem. *Consider a LPP (P) and its dual (D). If either problem is unbounded, then the other is infeasible. If either problem has an optimal solution, then so does the other, and the optimal values of the objective functions for the two problems are the same. Finally, any optimal solutions of (P) and (D) satisfy complementary slackness.*

7. More on Unboundedness and Infeasibility

The unboundedness criterion is that some entry of the objective row is negative, and the column above it contains no positive entry. From our initial study of the simplex method, we know that

> If a **feasible** tableau of a LPP satisfies the unboundedness criterion, then the LPP is unbounded.

If the unboundedness criterion holds in $T_{\boldsymbol{X}}$, then $T_{\boldsymbol{W}}$ satisfies the "infeasibility criterion": some entry of the right-hand column is negative, and the row to its left contains no negative entry in a column corresponding to a nonbasic variable.

When a tableau satisfies this infeasibility criterion, **all** the entries of the row in question are nonnegative; in the basic columns we see only 0's and 1. Therefore that row represents a constraint (C) with all nonnegative coefficients, but a negative right side. This constraint (C) cannot be satisfied by any nonnegative values of the variables. So there cannot be any feasible solution of underlying LPP.

> If **any** tableau of a LPP satisfies the infeasibility criterion, then the LPP is infeasible, i.e., has no feasible solution.

This explains again why the dual of an unbounded LPP is infeasible. (See Corollary 1 to the Weak Duality Theorem.)

But notice the lack of symmetry: for unboundedness we need a **feasible** tableau satisfying the U.C., but for infeasibility any tableau satisfying the I.C. will do. For this reason it is possible that the dual of an infeasible problem may be infeasible as well.

The text gives a nice chart of the possible outcomes for a primal and dual problem.

8. The optimal values of the dual variables are in the optimal primal tableau

Indeed, by (1) above, the value of a basic variable in the dual problem is the corresponding entry of the objective row of the primal tableau. The non-basic variables are always 0 at a basic solution.

Thus if we have solved (P) by reaching a tableau T satisfying the feasibility and optimality criteria, then the solution of (D) can be read off from T. The optimal value of the objective functions for (P) and (D) are the same. The values of the dual variables $w_1, \ldots, w_m$ can be found as follows. The nonbasic ones are of course 0, and the basic ones are in the objective row of the final (P)-tableau.

9. The Dual Simplex Method

A couple of remarks here. The ordinary ("primal") simplex method and dual simplex method apply to different kinds of starting tableaux for a LPP. Both aim to find an optimal solution of the LPP, but if no optimal solution exists, the two methods fail in different ways.

SIMPLEX:	any feasible tableau	$\longrightarrow$	an optimal solution, or LPP is unbounded
DUAL SIMPLEX:	any dual-feasible tableau	$\longrightarrow$	an optimal solution, or LPP is infeasible

When one carries out the dual simplex method on tableaux $T_{\boldsymbol{X}}$ for a LPP, the corresponding tableaux $T_{\boldsymbol{W}}$ for the dual problem are going through the steps of the simplex method.

10. Exercises

Ex. 1–2: Duality

1. For a certain LPP in standard form, slack variables have been introduced and the simplex method is applied. At a certain point the following tableau is reached. What is the tableau for the corresponding basic solution of the dual problem? What are the current values of the dual variables w_1, w_2, w_3? Is the current basic solution optimal? optimal for the dual? feasible? feasible for the dual?

	x_1	x_2	x_3	x_4	w_1^*	w_2^*	w_3^*	
x_2	5	1	0	−7	1	0	−2	4
x_3	−1	0	1	7	4	0	6	2
w_2^*	0	0	0	−3	17	1	8	21
	18	0	0	−11	26	0	18	34

2. From the tableau in problem 1, recover the original constraints $A\boldsymbol{x} \leq \boldsymbol{b}$ and cost vector $\boldsymbol{c}$.

Ex. 3–8: Properties 2 and 3

Problems 3–8 refer to a standard LPP: maximize $\boldsymbol{c}^T\boldsymbol{x}$ subject to $A\boldsymbol{x} \leq \boldsymbol{b}$ and $\boldsymbol{x} \geq 0$, and its dual problem. Both have had slack variables introduced. The problems are about tableaux $T_{\boldsymbol{X}}$ and $T_{\boldsymbol{W}}$ of the respective problems at corresponding basic solutions. As in Section 2, $T_{\boldsymbol{W}}$ is assumed to have its slack variables listed first across the top. Let $T^c_{\boldsymbol{X}}$ and $T^c_{\boldsymbol{W}}$ be the parts of the two tableaux in the inner box. We know from Chapter 3 that $T^c_{\boldsymbol{X}} = B[A|I]$ and $T^c_{\boldsymbol{W}} = B'[I| - A^T]$ for some invertible matrices B and B'. Moreover the resource columns in $T_{\boldsymbol{X}}$ and $T_{\boldsymbol{W}}$ are $B\boldsymbol{b}$ and $-B'\boldsymbol{c}$, respectively.

3. (PROPERTY 3c) Show that $T^c_{\boldsymbol{X}}(T^c_{\boldsymbol{W}})^T = \boldsymbol{0}$ (this isn't so bad; see the previous paragraph). Deduce that if g be a basic variable for (P) and h is a nonbasic variable for (P), then the entry of a of $T_{\boldsymbol{X}}$ in row g and column h is the negative of the entry b of $T^c_{\boldsymbol{W}}$ in row h' and column g'. (Hint. Show that the entry of $T^c_{\boldsymbol{X}}(T^c_{\boldsymbol{W}})^T$ in row g and column h' is $a + b$, using the fact that $T_{\boldsymbol{X}}$ and $T_{\boldsymbol{W}}$ have identity submatrices in their basic columns.)

4. Let $\boldsymbol{f}$ be the objective row of $T_{\boldsymbol{X}}$, without the lower right corner. Then show that the column $T^c_{\boldsymbol{W}}\boldsymbol{f}^T$ contains the nonbasic entries of $\boldsymbol{f}$, in the order in which the rows of $T_{\boldsymbol{W}}$ occur. (Hint. There's an identity submatrix of $T^c_{\boldsymbol{W}}$ "facing" the nonzero entries of $\boldsymbol{f}^T$.)

5. (PROPERTY 3b) Because pivots change the objective row only by adding multiples of constraint rows, we know that there are scalars $e_1, \ldots, e_m$ such that the objective row of T (excluding lower right corner) is

$$\boldsymbol{f} = [e_1 \ \ldots \ e_m \ 1] \begin{bmatrix} A & I \\ -\boldsymbol{c}^T & \boldsymbol{0}^T \end{bmatrix}.$$

Show that as a result, $-B'\boldsymbol{c} = T^c_{\boldsymbol{W}}\boldsymbol{f}^T$, justifying property 3b (see previous problem).

6. (PROPERTY 3a) Show that $T_{\boldsymbol{X}}$ and $T_{\boldsymbol{W}}$ have opposite objective function values. (Use the fact–coming from the objective row trick–that the w-portion of the objective row of $T_{\boldsymbol{X}}$ is $\boldsymbol{c}_B^T B^{-1}$ and the resource column is $\boldsymbol{r} = B^{-1}\boldsymbol{b}$. The objective function value is $\boldsymbol{c}_B^T\boldsymbol{r}$.)

7. (THE BASIS DETERMINES THE TABLEAU) Suppose that a matrix A' has been obtained from a matrix A by a sequence of row operations. Suppose also that A and A' are $m \times n$ and contain the columns of the $m \times m$ identity matrix **in the same positions**. Show that $A = A'$. (**Hint.** Each row of A' is a linear combination of the rows of A.)

8. (PROPERTY 2) Suppose that A is $m \times n$, and some set of m columns of $[A \,|\, I]$ has been selected. If this set of m columns is linearly independent, show that the "complementary" set of n columns of $[-I \,|\, A^T]$ is also linearly independent. Then explain why this problem and the last problem imply Property 2.