

The Primal-Dual Simplex Method

A Parametric Method
Math 354, Spring 2006

With the simplex method and the dual simplex method, we can solve any LPP as long as we can find a starting point—a basic solution which is feasible for either the primal or the dual problem. In the two-phase method, the goal of the first phase is to obtain such a starting point for the second phase.

A more unified method is the parametric self-dual simplex method or “primal-dual” method. The idea is to **begin by altering the LPP** to one with an obvious initial tableau which is feasible and dual-feasible¹, so is optimal. The altered problem is obtained from the original problem by increasing or decreasing certain coefficients by a large but unspecified number μ . The focus then shifts to getting rid of the parameter μ without losing optimality. Getting rid of it means making $\mu = 0$. We keep μ in the tableau **symbolically**, i.e., as a variable, and we imagine it taking on various values at various stages of the algorithm. At first, we imagine μ to be very large, and the goal is to reduce μ to $\mu = 0$ without giving up optimality. Then we will have an optimal solution of the original LPP.

We give an example² below, and discuss it there in some detail. But first, here is the plan. The LPP translates to Tableau 0, which is neither feasible nor dual-feasible. The parametrized problem begins with Tableau 1, which is feasible and dual-feasible if μ is large enough.

So imagine that μ starts out very large, and then shrinks toward 0, slowly. As μ shrinks, the tableau may suddenly stop being feasible or dual-feasible when μ crosses certain “threshold” values. When this happens, we immediately pause to “fix” the tableau. If we lost dual-feasibility (resp. feasibility) we restore it by the simplex method (resp. the dual simplex method). Having done that, we have a tableau which is feasible and dual-feasible for smaller values of μ than before. It will not be feasible and dual-feasible for very large μ any more, but we don’t care because our goal is $\mu = 0$. See Tableau 2 below.

Now let μ resume shrinking toward 0, and whenever the tableau loses feasibility or dual-feasibility, pause to “repair” the tableau. As there are only finitely many possible sets of basic variables, there are only finitely

¹We continue to use the term “dual-feasible” for a tableau satisfying the optimality criterion.

²The example is taken from Robert J. Vanderbei, *Linear Programming, Foundations and Extensions*, 2nd Edition, Springer, 2001. A highly recommended book.

many possible “threshold” values that μ can cross. Therefore, after finitely many repairs, this procedure finally reaches a tableau that is feasible and dual-feasible for $\mu = 0$, i.e., for the original problem, as in Tableaux 3, 4, 5. Of course there is also the possibility that before reaching $\mu = 0$, we reach a tableau showing that for some $\mu > 0$, the perturbed problem is infeasible or unbounded. Then we stop and can conclude that the original LPP or its dual was infeasible (exercise).

While we are reducing μ we “freeze” the basis until we cross a threshold value of μ . Then, we pause and freeze μ , and use simplex or dual-simplex to change the basis. Then we re-freeze the basis and reduce μ , and so on. Throughout, as μ changes, so does the LPP that the tableau represents.³

THE EXAMPLE.

Maximize $z = -2x + 3y$

subject to

$$\begin{aligned} -x + y &\leq -1 \\ -x - 2y &\leq -2 \\ y &\leq 1 \\ x, y &\geq 0 \end{aligned}$$

	x	y	u	v	w
u	-1	1	1	0	0
v	-1	-2	0	1	0
w	0	1	0	0	1
	2	-3	0	0	0

TABLEAU 0: original LPP

By adding slack variables we get Tableau 0. By duality theory, we want a tableau that is both feasible and dual-feasible. Our Tableau 0 is neither. We “perturb” the original problem, adding μ to each right side, and subtracting μ from each non-slack coefficient of the objective function. Thus μ is **added** to the objective row entries in Tableau 1. Each value of μ corresponds to a different LPP. (Our original problem corresponds to $\mu = 0$.)

	x	y	u	v	w
u	-1	1	1	0	0
v	-1	-2	0	1	0
w	0	1	0	0	1
	$(2 + \mu)$	$(-3 + \mu)$	0	0	0

TABLEAU 1: feasible and dual-feas. for $\mu \geq 3$

³This is an example of a general technique sometimes called “deformation”, where we introduce a parameter μ into a given problem, making a family of problems, one for each value of μ . One tries to do this in a way that (a) the problem is easy (or easier) for some value(s) of μ , and (b) one can keep track of what happens to the solution as we let $\mu \rightarrow 0$.

Tableau 1 is the beginning of our analysis of these problems. Keep in mind that we really only care about $\mu = 0$. If μ is very large, Tableau 1 is clearly feasible and dual-feasible. Instead of specifying a value for μ , we observe that as long as $\mu \geq 3$, this tableau is feasible and dual-feasible. We will reduce the perturbation factor μ to 0, gradually, repairing any loss of feasibility or dual-feasibility as soon as it happens.

You should try to spot the threshold values of μ from the tableaux, and use the text to check yourself. What is the threshold for Tableau 1? We just watch for values of μ where one of $2 + \mu$, $-3 + \mu$, $-1 + \mu$, $-2 + \mu$, or $1 + \mu$ changes sign. As μ descends, the first such threshold is $\mu = 3$. For values of μ slightly less than $\mu = 3$, Tableau 1 has lost dual-feasibility: the $-3 + \mu$ in the objective row becomes negative. However, the tableau is still feasible for such values of μ . Therefore, thinking of μ as slightly less than 3, we perform primal simplex iterations. The only choice for entering variable is y . The ratios $(-1 + \mu)/1$ and $(1 + \mu)/1$ compete to determine the departing variable. The first of these is the smaller, and so u departs. Pivoting on the 1 in the u -row, y -column brings us to Tableau 2.

For Tableau 2 to be feasible and dual-feas., the conditions are that $-1 + 2\mu$, $3 - \mu$, $-1 + \mu$, and $-4 + 3\mu$ must all be nonnegative. These conditions all hold only for μ between $\frac{4}{3}$ and 3. Next threshold: $\mu = \frac{4}{3}$. It is convenient to ignore the lower right corner until the final tableau.

Just below $\mu = 4/3$ Tableau 2 is infeasible in the v -row, but remains dual-feasible.

So the dual simplex method tells us to let v depart and x enter.

After pivoting on the -3 , we have reached Tableau 3.

As μ drops below $4/3$, feasibility holds all the way to $\mu = 0$ in Tableau 3, but dual-feasibility fails if $2\mu - 1 < 0$. The next threshold is $\mu = 1/2$.

	x	y	u	v	w
y	-1	1	1	0	0
v	-3	0	2	1	0
w	1	0	-1	0	1
	$(2\mu - 1)$	0	$(3 - \mu)$	0	0

TABLEAU 2: *feas. and dual-feas. if $\frac{4}{3} \leq \mu \leq 3$*

	x	y	u	v	w
y	0	1	$1/3$	$-1/3$	0
x	1	0	$-2/3$	$-1/3$	0
w	0	0	$-1/3$	$1/3$	1
	0	0	$(7 + \mu)/3$	$(2\mu - 1)/3$	0

TABLEAU 3: *feas. and dual-feas. if $\frac{1}{2} \leq \mu \leq \frac{4}{3}$*

	x	y	u	v	w
y	0	1	0	0	1
x	1	0	-1	0	1
v	0	0	-1	1	3
	0	0	$2 + \mu$	0	$1 - 2\mu$

TABLEAU 4: *feasible and dual-feasible if $0 \leq \mu \leq 1/2$*

	x	y	u	v	w
y	0	1	0	0	1
x	1	0	-1	0	1
v	0	0	-1	1	3
	0	0	2	0	1

TABLEAU 5: *optimal for the original LPP*

Next imagine μ slightly below $1/2$ and make a (primal) pivot in Tableau 3, with v entering, w departing. This leads to Tableau 4.

Tableau 4 is feasible and dual-feasible even for $\mu = 0$, so we have achieved our objective. Explicitly, set $\mu = 0$ to get Tableau 5, which is the final tableau for the original LPP. Now, compute the optimal value of z . Our optimal solution is $x = 2$, $y = 1$, $z = -1$.

(Shortcuts for hand calculations. It is not really necessary to go to the trouble of writing down all of Tableau 5 since it's implicit in Tableau 4, except for the value of z . Also, we actually could have set $\mu = 0$ in Tableau 3. The result isn't dual-feasible, but it's at least feasible, and then we can use primal simplex method.)

Exercises.

- Solve some LPP's by the primal-dual method.
- In the primal-dual method, assuming that $\mu \geq 0$, any feasible solution of the original LPP is also feasible for the perturbed problem. Explain why this is true. (The same is true for dual-feasibility, too.)
- In The Example, we were always able to restore feasibility and dual-feasibility. But it is possible in principle that in the course of the primal-dual method, the simplex or dual-simplex method might lead to the unboundedness or infeasibility criterion for the perturbed problem. What would that mean about the original LPP and its dual? (Hint: Use the previous exercise.)
- Another conceivable difficulty is that both feasibility and dual-feasibility might be lost at the **same** threshold value of μ . Once both are lost, we can't restore either one, at least by the simplex or dual-simplex methods. Can you think of a slightly different way to deform the original LPP that prevents this unpleasant possibility from ever occurring?
- In The Example, as we crossed each threshold value of μ , it took just one iteration of simplex or dual-simplex to restore feasibility and dual-feasibility. Do you think that this always necessarily true? If so, why? If several iterations might be needed, how do you tell when to freeze the basis and start reducing μ again?