

Supplementary notes on tableaux

Math 354-04, Spring 2006

These notes contain some basic facts about tableaux, the optimality criterion, and the objective row. Consider a LPP in canonical form:

$$\begin{aligned} & \text{Maximize} && z = c^T x = c_1 x_1 + \dots + c_n x_n \\ & \text{subject to} && Ax = b \\ & && x \geq 0, \quad \text{where } x = [x_1 \dots x_n]^T. \end{aligned}$$

The initial setup S for this problem is at the right. The objective row stands for

$$z - \sum c_k x_k = 0.$$

$$S =$$

	x_1	x_2	\dots	x_k	\dots	x_n	
	A						
	$-c_1$	$-c_2$	\dots	$-c_k$	\dots	$-c_n$	0
							b

The other rows, called “constraint rows”, represent the constraints $Ax = b$.

Warning. S may or may not be a tableau, and frequently it is not even written down. It is here as an aid to understanding what’s going on.

- 1 • Remember that any vector x satisfying $Ax = b$ is called a “solution” of our LPP. Any solution of our LPP satisfies all the equations represented by S .
- 2 • To each basic solution, i.e., each choice of m of the variables x_1, \dots, x_n which shall be considered “basic”, there corresponds one and only one¹ “tableau”:

$$T =$$

	x_1	x_2	\dots	x_j	\dots	x_n	
	A'						
							b'
	f_1	f_2	\dots	f_k	\dots	f_n	z_0

$$=$$

	x_1	x_2	\dots	x_k	\dots	x_n	
	$:$	$:$	$:$	$:$	$:$	$:$	$:$
	x_{k_j}	a'_{j1}	a'_{j2}	\dots	a'_{jk}	\dots	a'_{jn}
	$:$	$:$	$:$	$:$	$:$	$:$	$:$
	f_1	f_2	\dots	f_k	\dots	f_n	z_0

¹ Except for the order in which the constraint rows appear. The conditions that make T unique (with this exception) are 3 and 4 below.

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SUPPLEMENTARY NOTES: TABLEAUX

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All the decision variables x_1, \dots, x_n are written across the top of T , and all the basic variables x_{k_1}, \dots, x_{k_m} are listed down the left side.

- 3 • The tableau T must be obtainable from the underlying setup S by pivoting on entries in the central portion (where the A and A' are). Often S itself is a tableau, but not always. In any case the simplex method starts with an initial tableau easily traceable by row operations back to the initial setup S . The method then proceeds, one pivot at a time, to further tableaux.
- 4 • The columns of T under the basic variables must be the columns of the $m \times m$ identity matrix², in some order, with 0’s beneath them in the objective row. The list of basic variables at the left indicates where the 1’s are in the columns.
- 5 • Because any tableau T arises from S by pivoting (see 3 above), the rows of $[A' | b']$ are linear combinations of the rows of $[A | b]$, and vice-versa since row operations are reversible. Therefore the system of equations $A'x = b'$ has the same solutions as $Ax = b$. The “solutions” of our LPP can be characterized not only as the solutions of $Ax = b$, but also as the solutions of the system $A'x = b'$. Again because of the pivoting, the objective row of T is the same as that of S , plus some linear combination of the rows of $[A | b]$, or equivalently, plus a linear combination of the rows of $[A' | b']$. Therefore the equations represented by the rows of T hold for any solution x of our LPP.
- 6 • The tableau T corresponds to the basic solution

$$x_{k_1} = b'_1, \dots, x_{k_m} = b'_m; \text{ all nonbasic } x_k \text{ equal } 0; \text{ and } z = z_0.$$

This basic solution is **feasible** if and only if $b' \geq 0$.

When we choose which variables are basic, we are choosing a point of view from which to analyze the LPP, and the tableau is the “picture” of the LPP from that point of view.

Some people prefer to write the equations instead of the tableau, and call the equations a “dictionary” for that choice of basic variables.

²As we know, the columns of A corresponding to the basic variables must be linearly independent. Hence it is possible to reach this identity submatrix by pivoting.

- 7 • Let's look at an example both ways. We start with Tableau 1. Then v enters the basis, y departs, and the pivot on 0.6 gives Tableau 2. Then u enters, x departs, and pivoting on 2/3 gives Tableau 3.

	x	y	u	v	
u	0.2	0.6	1	0	3000
v	0.8	0.4	0	1	5000
	-1	-2	0	0	0

	x	y	u	v	
y	1/3	1	5/3	0	5000
v	2/3	0	-2/3	1	3000
	-1/3	0	10/3	0	10000

Tableau 1: x, y nonbasicTableau 2: x, u nonbasic

x, y Dictionary:

$$u = -(0.2)x - (0.6)y + 3000$$

$$v = -(0.8)x - (0.4)y + 5000$$

$$z = x + 2y$$

x, u Dictionary:

$$y = -(1/3)x - (5/3)u + 5000$$

$$v = -(2/3)x + (2/3)u + 3000$$

$$z = (1/3)x - (10/3)u + 10000$$

All these tableaux represent basic feasible solutions, since each right hand column ≥ 0 (it is not necessary to check that $z_0 \geq 0$). In each case the z -equation, valid on the whole feasible region, has the form

$$z = z_0 - \sum_{i=1}^n f_i x_i$$

where f_i is the objective row entry for the variable x_i . The basic variables don't contribute to the sum, since $f_i = 0$ for the basic variables.

At the current basic feasible solution $z = z_0$ since the non-basic variables are 0 there. But the z -equation also tells us how z changes if we move away from this basic feasible solution, with one or more nonbasic variables changing (increasing! to maintain feasibility).

In Tableau 1, the x and y coefficients in the z -equation are both positive, so we can increase z by increasing x or y . We chose y to bring into the

	x	y	u	v	
y	0	1	2	-5	3500
x	1	0	-1	1.5	4500
	0	0	3	0.5	11500

Tableau 3: u, v nonbasic

u, v Dictionary:

$$y = -2u + (0.5)v + 3500$$

$$x = u - (1.5)v + 4500$$

$$z = -3u - (0.5)v + 11500$$

basis, and got Tableau 2. (The simplex method told us that u had to be the departing variable.) Similarly, in Tableau 2, the z -equation shows that z can be increased by increasing x . In Tableau 3, the z -equation now shows that z can't be increased by making any feasible change in any of the variables. So Tableau 3 gives an optimal solution to the LPP.

- 8 • What made Tableau 3 optimal? Answer: there were no positive coefficients in the z -equation. Equivalently all the entries of the objective row of the tableau were ≥ 0 (ignoring z_0 , which doesn't matter here). This in general is the OPTIMALITY CRITERION for a tableau T :

The objective row entries f_i satisfy $f_i \geq 0$ for all $i = 1, \dots, n$,

and it ensures optimality since T tells us that

$$z = z_0 - \sum f_i x_i \quad \text{on the entire feasible region.}$$

- 9 • **Objective row trick.** If you have a complete tableau calculated except for the objective row, here is a trick for calculating the objective row, without pivoting or even knowing what the objective row was in the previous tableau. (Or if you do pivot, use the trick to check your arithmetic.) As an illustration, Tableau 2 is at right without its objective row, which we will recover. Write the coefficients c_1, \dots, c_n of the objective function above T . Put the coefficients of just the basic variables c_{k_1}, \dots, c_{k_m} down the left side of T , calling that column vector c_B . Let A'_1, \dots, A'_m be the columns of A' . The trick is:

$$(1) \quad \text{each } f_k = -c_k + c_B \bullet A'_k = -c_k + \sum_{j=1}^m c_{k_j} a'_{jk} \quad \text{and} \quad z_0 = c_B \bullet b'.$$

For Tableau 2, $c_B = [2 \ 0]^T$, $z_0 = 2 \cdot 5000 + 0 \cdot 3000 = 10000$,

$$f_1 = -1 + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} = -1/3 \quad \text{and} \quad f_3 = -0 + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 5/3 \\ -2/3 \end{bmatrix} = 10/3$$

Of course the basic entries of the objective row (f_2 and f_4 in this case) are necessarily 0. This recovers the objective row we had in **7**.

10 • Justification for the trick. Denote by $\mathbf{R}_1, \dots, \mathbf{R}_m$ the rows of $[A' | \mathbf{b}]$. All the equations in (1) make a single vector equation for the objective row:

$$(2) \quad [f_1 \cdots f_n \ z_0] = [-c_1 \cdots -c_n \ 0] + \sum_{j=1}^m c_{k_j} \mathbf{R}_j.$$

We know from **4** that some such equation is true, but not necessarily with these coefficients c_{k_1}, \dots, c_{k_m} . Let's show that the c_{k_j} really are the right coefficients.

Pick any $j = 1, \dots, m$. We know that $f_{k_j} = 0$ since x_{k_j} is basic. Looking down the x_{k_j} column we see 1 in row \mathbf{R}_j and 0 in all the other \mathbf{R}_i 's. So the correct version of equation (2) tells us that

$$0 = f_{k_j} = -c_{k_j} + \text{correct coefficient of } \mathbf{R}_j.$$

So the correct coefficient is c_{k_j} , (2) is correct as it stands, and (1) is true.

11 • THE UNBOUNDEDNESS CRITERION: An LPP is unbounded if a tableau T is reached representing a basic **feasible** solution, such that $f_k < 0$ for some k , and $a_{kj} \leq 0$ for all $j = 1, \dots, m$ (these are the entries of the column A'_k directly above f_k). Why is this? Remember the dictionary for T , with only nonbasic x 's on the right side:

$$\begin{aligned} x_{k_1} &= b'_{k_1} - \cdots - a_{k_1 1} x_1 - \cdots \\ &\dots \end{aligned}$$

$$\begin{aligned} x_{k_m} &= b'_{k_m} - \cdots - a_{k_m} x_k - \cdots \\ z &= z_0 - \cdots - f_k x_k - \cdots \end{aligned}$$

Since this basic solution is feasible, all the b 's are nonnegative. Now send \mathbf{x} on a journey through the feasible region, moving it away from this basic feasible solution and increasing x_k to $+\infty$ for the single x_k shown in the above equations. We can keep \mathbf{x} feasible as it travels by adjusting its other coordinates as follows. All the nonbasic variables except x_k stay equal to 0. As a result, the dots in the above equations amount to 0 and can be ignored. We change the basic x_{k_j} 's according to the first bunch of equations. Then $x_{k_j} = b'_{k_j} - a_{k_j} x_k \geq 0$, so \mathbf{x} stays a feasible solution (Recall: all $a_{k_j} \leq 0$). Finally $z = z_0 - f_k x_k \rightarrow +\infty$, since $f_k < 0$. So the LPP is unbounded.