

1. The first equation must be multiplied through by -1 , and takes a surplus variable and an artificial variable. The second equation takes an artificial variable. The objective function in Phase 1 is $z = -y_1 - y_2$. The objective row trick can be used to get the initial objective row. Values marked X don't matter.

| | x_1 | x_2 | x_3 | u_1 | y_1 | y_2 | |
|-------|-------|-------|-------|-------|-------|-------|-----|
| y_1 | -1 | 2 | -8 | -1 | 1 | 0 | 12 |
| y_2 | 1 | -1 | 5 | 0 | 0 | 1 | 2 |
| | 0 | -1 | 3 | 1 | 0 | 0 | -14 |

\longrightarrow
 2 pivots

| | x_1 | x_2 | x_3 | u_1 | y_1 | y_2 | |
|-------|-------|-------|-------|-------|-------|-------|----|
| x_2 | 1.5 | 1 | 0 | -2.5 | X | X | 8 |
| x_3 | 0.5 | 0 | 1 | -0.5 | X | X | 38 |
| | X | X | X | X | X | X | 0 |

This is the end of Phase 1; no artificial variables remain in the basis. Use the objective row trick to start Phase 2, with $z' = -z = -x_1 - 2x_2 - x_3$.

| | x_1 | x_2 | x_3 | u_1 | |
|---------|-------|-------|-------|-------|-----|
| $-2x_2$ | 1.5 | 1 | 0 | -2.5 | 38 |
| $-1x_3$ | 0.5 | 0 | 1 | -0.5 | 8 |
| | -2.5 | 0 | 0 | 5.5 | -84 |

\longrightarrow
 1 pivot

| | x_1 | x_2 | x_3 | u_1 | |
|---------|-------|-------|-------|-------|-----|
| $-2x_2$ | 0 | 1 | -3 | -1 | 14 |
| $-1x_1$ | 1 | 0 | 2 | -1 | 16 |
| | 0 | 0 | 5 | 3 | -44 |

The optimal solution is $x_1 = 16$, $x_2 = 14$, $x_3 = u_1 = 0$, $z_{opt} = -z'_{opt} = 44$.

2. For standard form we have to maximize $z' = -z$, we have to write $x_2 = x_2^+ - x_2^-$ everywhere, and we have to multiply the middle constraint by -1 to bring it to \leq form. The 2-phase method is not needed because the right sides (in standard form) are all nonnegative.

Maximize $z' = -7x_1 - 2x_2^+ + 2x_2^- + 4x_3$

subject to

$$x_2^+ - x_2^- + x_3 \leq 6$$

$$-x_1 + 2x_3 \leq 3$$

$$-5x_1 + 12x_2^+ - 12x_2^- \leq 5$$

$$x_1 \geq 0, x_2^+ \geq 0, x_2^- \geq 0, x_3 \geq 0$$

| | x_1 | x_2^+ | x_2^- | x_3 | u_1 | u_2 | u_3 | |
|-------|-------|---------|---------|-------|-------|-------|-------|---|
| u_1 | 0 | 1 | -1 | 1 | 1 | 0 | 0 | 6 |
| u_2 | -1 | 0 | 0 | 2 | 0 | 1 | 0 | 3 |
| u_3 | -5 | 12 | -12 | 0 | 0 | 0 | 1 | 5 |
| | 7 | 2 | -2 | -4 | 0 | 0 | 0 | 0 |

(b) The x_2^- column shows that this LPP is unbounded.

3. See notes or text.
4. Phase 1 ends with a strictly negative optimal value for z .
5. (a) $z + 10x_3 + 18x_4 + 16x_6 = 270$, for any feasible solution \mathbf{x} . (b) Only the objective row changes. Use the objective row trick. The new objective row: 0 0 50 88 0 66 1270
6. Since optimality criterion (dual feasibility) holds, use the dual simplex method. The departing variable is x_2 and the θ ratios $|10/(-2)| = 5$ and $|16/(-1)| = 16$ show that the pivot should be the -2 in the top row; x_3 enters. One pivot gets to optimality. The new optimal solution is $x_1 = 10$, $x_3 = 10$, $x_5 = 120$, $x_2 = x_4 = x_6 = 0$, $z_{opt} = 70$.

| | x_1 | x_2 | x_3 | x_4 | x_5 | x_6 | |
|-------|-------|-------|-------|-------|-------|-------|-----|
| x_3 | 0 | -0.5 | 1 | -1.5 | 0 | 0.5 | 10 |
| x_1 | 1 | 0 | 0 | 13 | 0 | 3 | 10 |
| x_5 | 0 | -5 | 0 | -17 | 1 | 17 | 120 |
| | 0 | 5 | 0 | 33 | 0 | 11 | 70 |

7. Since it's a max problem, consider it as a "modified" standard form LPP. The dual is then a "modified" dual standard form LPP. The modifications are:

first primal constraint is \geq , implying
 first dual variable ≤ 0
 x_2 unconstrained, implying
 second dual constraint is =

The dual problem is
 Minimize $w = 200w_1 + 100w_2$
 subject to

$$\begin{aligned} 5w_1 + 7w_2 &\geq 2 \\ 12w_1 - 2w_2 &= -3 \\ w_1 &\leq 0, \quad w_2 \geq 0 \end{aligned}$$

Many other answers are also correct.

8. (a) Only $[0 \ 20 \ 0 \ -30 \ 0]^T$ is basic. $[70 \ 10 \ 0]^T$ is not basic because the corresponding (1st and 3rd) columns of A are not linearly independent. Of the other two, one is not a solution and the other has too few zeros.

(b) None. Reason: none of them are basic feasible solutions.

9. (a) Suppose that \mathbf{u} and \mathbf{v} are unequal optimal solutions. Let z be the objective function. Then $z(\mathbf{u}) = z_{opt} = z(\mathbf{v})$. Since z is a linear function and $\mathbf{u} \neq \mathbf{v}$, z must be constant on the whole line joining \mathbf{u} and \mathbf{v} . Now let \mathbf{x} be any point on the line segment from \mathbf{u} to \mathbf{v} . Since the feasible region is convex, \mathbf{x} is feasible, and we just showed that $z(\mathbf{x}) = z_{opt}$ so \mathbf{x} is optimal. Therefore the set of optimal solutions is convex.

(Note: There can be more than one optimal solution; for this to occur there must be a 0 somewhere in the objective row but not in a basic column. That is, the optimal solution to the dual must be degenerate.)

(b) The first array below is not a tableau, but is made a tableau by subtracting row 1 from row 4. Then one dual simplex pivot gets to the optimal solution.

| x_1 | x_2 | x_3 | x_4 | x_5 | x_6 | u_1 | |
|-------|-------|-------|-------|-------|-------|-------|-----|
| 0 | 1 | -2 | 3 | 0 | -1 | 0 | 30 |
| 1 | 0 | 4 | 7 | 0 | 5 | 0 | 100 |
| 0 | 0 | -10 | -2 | 1 | 12 | 0 | 70 |
| 0 | 1 | 0 | 2 | 0 | 0 | 1 | 20 |
| 0 | 0 | 10 | 18 | 0 | 16 | 0 | 270 |

\longrightarrow
 1 row op
 makes it a
 tableau

| x_1 | x_2 | x_3 | x_4 | x_5 | x_6 | u_1 | | |
|-------|-------|-------|-------|-------|-------|-------|---|-----|
| x_2 | 0 | 1 | -2 | 3 | 0 | -1 | 0 | 30 |
| x_1 | 1 | 0 | 4 | 7 | 0 | 5 | 0 | 100 |
| x_5 | 0 | 0 | -10 | -2 | 1 | 12 | 0 | 70 |
| u_1 | 0 | 0 | 2 | -1 | 0 | 1 | 1 | -10 |
| | 0 | 0 | 10 | 18 | 0 | 16 | 0 | 270 |

Pivot on -1 in last row (Note: if you calculate the resource column first, you see that it's feasible so the nonpivot columns don't really need to be calculated.)

| x_1 | x_2 | x_3 | x_4 | x_5 | x_6 | u_1 | | |
|-------|-------|-------|-------|-------|-------|-------|----|----|
| x_2 | 0 | 1 | 4 | 0 | 0 | 2 | 3 | 0 |
| x_1 | 1 | 0 | 18 | 0 | 0 | 12 | 7 | 30 |
| x_5 | 0 | 0 | -14 | 0 | 1 | 10 | -2 | 90 |
| x_4 | 0 | 0 | -2 | 1 | 0 | -1 | -1 | 10 |
| | 0 | 0 | 46 | 0 | 0 | 34 | 18 | 90 |

The optimal solution is $x_1 = 30$, $x_4 = 10$, $x_5 = 90$, $x_2 = x_3 = x_6 = u_1 = 0$, $z_{opt} = 90$.