DO SIX OF THE SEVEN PROBLEMS BELOW.

IF YOU WORK ON ALL SEVEN, INDICATE CLEARLY WHICH SIX PROBLEMS ARE TO BE GRADED BY CROSSING OUT YOUR WORK ON ONE OF THEM.

You may use any result from the book or from class notes, unless the result is essentially equivalent to what you are asked to prove or the problem instructs you otherwise. In quoting an established result it is not necessary to be extremely formal, but please make it clear that you know and have verified the necessary hypotheses. You may use abbreviations and symbols like iff, $\forall$, $\exists$, etc.

1. If α and β are real numbers (i.e., cuts), define what it means to have $\alpha < \beta$; you need not verify that your definition gives an order relation. Prove that the set $\mathbb{R}$ of all real numbers, with this order relation, has the least upper bound property.

2. Prove, directly from the definition of compactness, that every compact subset of $\mathbb{R}$ is closed and bounded.

3. Prove that the collection of all finite subsets of the natural numbers is countable.

4. Let S and T be nonempty sets of positive real numbers, each of which is bounded above, and let $A = \{st \mid s \in S,\, t \in T\}$. Prove that $\sup A = \sup S \cdot \sup T$.

5. Let G be a bounded open subset of $\mathbb{R}$, and define a relation on G by $x \sim y$ iff the closed interval with endpoints x and y is a subset of G. You may assume without proof that this is an equivalence relation. Prove that each equivalence class is an open interval and that the number of such classes is at most countable.

6. (a) Let X be a metric space with metric d, let K be a nonempty compact subset of X, and let x be a point of X with which does not belong to K. Prove that there exists a point $y \in K$ with $d(x, y) = \inf \{d(x, z) \mid z \in K\}$.

(b) Show by example that the conclusion need not hold if we assume only that K is nonempty and closed.

7. Let A and B be connected subspaces of a metric space X, and suppose that some point $a \in A$ is a limit point of B. Prove that $A \cup B$ is connected.