1. For r a rational number and α a cut, prove that $\alpha > r^*$ if and only if $r \in \alpha$.

2. Using the definitions of multiplication of cuts (real numbers) given in the text or equivalently in class, prove the following statements. Note that you will have to consider separately the cases $\alpha > 0^*$, $\alpha = 0^*$, and $\alpha < 0^*$.

(a) if α is a cut then $1^*\alpha = \alpha$.

(b) if α is a cut and $\alpha \neq 0^*$ then there is a cut $1/\alpha$ with $\alpha(1/\alpha) = 1^*$.

In problems 3 through 5 we use the following definition: if $\mathbb{F}_1$ and $\mathbb{F}_2$ are ordered fields, then $\mathbb{F}_1$ is an *ordered subfield* of $\mathbb{F}_2$ if $\mathbb{F}_1 \subset \mathbb{F}_2$ (as sets) and the (intrinsic) order and field operations on $\mathbb{F}_1$ are the same as the order and operations that $\mathbb{F}_1$ inherits from $\mathbb{F}_2$. Note that the results of these three problems imply that $\mathbb{R}$ is the unique ordered field with the least upper bound property.

3. Suppose that $\mathbb{F}$ is an ordered field. Prove that $\mathbb{Q}$ is an ordered subfield of $\mathbb{F}$. [More precisely, prove that there is a 1-1 mapping $\phi : \mathbb{Q} \to \mathbb{F}$ such that for all $x, y \in \mathbb{Q}$, $\phi(x+y) = \phi(x) + \phi(y)$, $\phi(xy) = \phi(x)\phi(y)$, and $\phi(x) < \phi(y)$ whenever $x < y$. Thus we may identify $\mathbb{Q}$ with the ordered subfield $\phi(\mathbb{Q})$ of $\mathbb{F}$.]

4. Suppose that $\mathbb{F}$ is an ordered field with the least upper bound property, and that $\mathbb{Q}$ is an ordered subfield of $\mathbb{F}$. Prove that $\mathbb{R}$ is also an ordered subfield of $\mathbb{F}$ (in the precise sense given in the previous problem).

5. Suppose that $\mathbb{F}$ is an ordered field with the least upper bound property, and that $\mathbb{R}$ is an ordered subfield of $\mathbb{F}$. Prove that $\mathbb{F} = \mathbb{R}$. Hint: Theorem 1.20 of Rudin was proved using only the fact that $\mathbb{R}$ is a complete ordered field with $\mathbb{Q}$ as an ordered subfield, so it continues to hold if $\mathbb{R}$ is replaced by $\mathbb{F}$.