Assignment 10

Due Wednesday, April 22

Exercises:

Chapter 9: 27, 28, 29, 30*, 31*

Chapter 10: 9*

Remarks, hints, and extra questions:

9.27, 9.28: These are examples showing that two of our results can indeed fail if the hypotheses are not satisfied. Compare with Theorems 9.41 and 9.42, respectively.

9.29: You need to understand this problem to do the next one, on Taylor series in several variables.

9.30 A long problem but I hope not too difficult. To be sure you see what is going on, take $n = 3$ and write out explicitly the Taylor approximation up to order 3, in the notation of part (c). There should be one term at order 0, three at order 1, six at order two, and ten at order 3. Don't worry about the remainder.

9.31 Rudin poses this problem in a rather vague way. Rather than try to guess what he means, do the following. The *Hessian* $H_{\mathbf{f}}(\mathbf{x})$ of a mapping $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\mathbf{x}$ is the $n \times n$ matrix formed by the second order partial derivatives $D_{ij}\mathbf{f}(\mathbf{x})$ of $\mathbf{f}$ at $\mathbf{x}$. (Many people wanted to use the Hessian on Exercise 9.21; if you are not familiar with it, this is what they were talking about.)

(a) Prove that a point $\mathbf{a}$ at which the gradient of $\mathbf{f}$ vanishes is a local minimum if $H_{\mathbf{f}}(\mathbf{a})$ is *positive definite*, which means that for any nonzero vector $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{u}^T H_{\mathbf{f}}(\mathbf{a})\,\mathbf{u} > 0$ ($\mathbf{u}$ is a column vector; $\mathbf{u}^T$ is its transpose, a row vector). Formulate a similar condition for a local maximum.

(b) Prove that 2×2 symmetric matrix $H = (h_{ij})_{i,j=1,2}$ is positive definite if and only if h_{11}, h_{22}, and $\det H$ are all positive. Hint: First consider the positivity condition for the two special vectors $\mathbf{u}^T = [1, 0]$ and $\mathbf{u}^T = [0, 1]$. Then for an arbitrary $\mathbf{u}$ write out $\mathbf{u}^T\, H\, \mathbf{u}$ explicitly in terms of u_1 and u_2, and complete the square.

10.9. A tough problem! Notice that in the formula we are to prove the right hand side is defined by the discussion in paragraphs 10.1 and 10.2 (the integral is over a 2-cell), and the left side by the discussion in paragraph 10.3 (if we interpret $\int_D$ as $\int_{\mathbb{R}^2}$). Again the hint is a little vague; here is one approach. Let I denote the rectangle $[0, a] \times [0, 2\pi]$ and I_0 its interior.

(i) Verify the mapping properties and the formula for the Jacobian as Rudin asks; check also that I_0 maps onto D_0. Then check that, as Rudin indicates, the formula holds if $\operatorname{supp} f \subset D_0$.

(ii) Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on I and satisfying $0 \le \phi_n \le 1$, $\operatorname{supp}\phi_n \subset I_0$, and $\lim_{n\to\infty} \phi_n = 1$ on I_0, with uniform convergence on compact subsets of I_0. For $(x, y) \in D$ with $(x, y) = T(r, \theta)$ let $\psi_n(x, y) = \phi_n(r, \theta)$. ($T$ is not 1-1 on D; why is ψ_n well defined?) Then show that

$$\lim_{n\to\infty} \int_0^a \int_0^{2\pi} \phi_n(r,\theta) f(T(r,\theta)) r \, d\theta \, dr = \int_0^a \int_0^{2\pi} f(T(r,\theta)) r \, d\theta \, dr, \tag{9.1}$$

and

$$\lim_{n\to\infty} \int_D \psi_n(x,y) f(x,y) \, dx \, dy = \int_D f(x,y) \, dx \, dy. \tag{9.2}$$

(iii) Construct a sequence ϕ_n as in (iii) and thus prove the theorem.

Note: (9.1) and (9.2) seem intuitively clear but perhaps tricky to prove carefully. In thinkng about (9.1) it may help to review problem 4(a) on our first exam; example 10.4 might also help. (9.2) is similar but the geometry is more complicated and I think that it will be very awkward to prove from our definitions of the integrals.

*Turn in starred problems Wednesday 4/22,