

## Assignment 11

This assignment will not be collected

**Exercises:**

Chapter 10: 10, 11

11.A Show that if  $\lambda$  is a 1-form in  $\mathbb{R}^n$  then  $\lambda \wedge \lambda = 0$ . Given  $\rho = \sum_{1 \leq i < j \leq 4} dx_i \wedge dx_j$ , a 2-form in  $\mathbb{R}^4$ , compute  $\rho \wedge \rho$  and thus verify that it is not zero. For what values of  $k$  must a  $k$ -form  $\omega$  satisfy  $\omega \wedge \omega = 0$ ?

11.B Let  $f = (x + yz)$ ,  $\lambda = xy dx + z^2 dy + 3 dz$ ,  $\rho = z dy + (x^2 + y^2 + z^2) dz$ , and  $\omega = -2 dx \wedge dy + y dx \wedge dz + y^2 z dy \wedge dz$  be differential forms in  $\mathbb{R}^3$ .

(i) Compute  $f \wedge \lambda$ ,  $\lambda \wedge \lambda$ ,  $\lambda \wedge \rho$ ,  $\lambda \wedge \omega$ , and  $\omega \wedge \omega$ , or, in two cases (which two?) observe without any computation that the product in question must be zero.

(ii) Compute  $df$ ,  $d\lambda$ ,  $d\rho$ , and  $d\omega$ . Compute  $d(d\lambda)$  and verify that it is zero.

(iii) Suppose now that  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is the linear map with matrix  $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ . Compute  $f_T$ ,  $\lambda_T$ , and  $\omega_T$ . Compute  $(d\lambda)_T$  and  $d(\lambda_T)$  and thus verify that they are equal.

(iv) Suppose that the singular  $k$ -cubes  $\varphi_k : J_k \rightarrow \mathbb{R}^3$  are defined by  $\varphi_0 = (1, 2, 3)$ ,  $\varphi_1(s) = (\cos \pi s, \sin \pi s, 0)$ ,  $\varphi_2(s, t) = (s, s, t)$ , and  $\varphi_3(s, t, u) = (s, 2t, 3u)$ . Describe these  $k$ -cubes geometrically. Evaluate  $\int_{\varphi_0} f$ ,  $\int_{\varphi_1} \lambda$ ,  $\int_{\varphi_2} \omega$ , and  $\int_{\varphi_3} dy \wedge dx \wedge dz$ .

11.C (Spivak) Let  $E \subset \mathbb{R}^n$  be open, let  $\varphi$  be a singular  $k$ -cube in  $E$ , and let  $T : J^k \rightarrow J^k$  be a 1-1, onto,  $C^\infty$  mapping. Show that if  $\omega$  is a  $k$ -form in  $E$  then  $\int_{\varphi} \omega = \int_{\varphi \circ T} \omega$ . **Note:** there are technical difficulties here of the sort we encountered in Exercise 10.9, arising from the fact that Theorem 10.9 applies to continuous functions (of compact support) defined in all of  $\mathbb{R}^n$ , not to functions defined on  $k$ -cells. Ignore these difficulties: assume that Theorem 10.9 applies if  $T$  is the bijective mapping from  $J^k$  to  $J^k$  considered in the problem.

11.D A  $k$ -form  $\omega$  is called *closed* if  $d\omega = 0$  and *exact* if  $\omega = d\kappa$  for some  $(k - 1)$ -form  $\kappa$ . Why is every exact form closed? Show that the 1-form

$$\lambda = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

in  $\mathbb{R}^2 \setminus \{(0, 0)\}$  is closed but not exact. Hint: consider  $\int_{\phi} \lambda$ , where  $\phi$  is the 1-cube  $\phi(t) = (\cos(2\pi t), \sin(2\pi t))$ . As Spivak points out, it is natural to call  $\lambda$  “ $d\theta$ ”. Why? Why is this not really mathematically correct?

11.E Verify the identity

$$d\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} = \frac{\partial(x_{i_1}, \dots, x_{i_k})}{\partial(s_1, \dots, s_k)} ds_1 \wedge \cdots \wedge ds_k \quad (10.1)$$

discussed in the notes (equation (11)).