Assignment 3

Due Wednesday, February 11

Exercises:

Chapter 7: 20*, 21*, 22*, 23, 24*

3.A **Extra credit.** A function f defined on $\mathbb{R}$ is called *periodic with period* T if $f(x+T) = f(x)$ for all $x \in \mathbb{R}$. Let $\mathcal{C}_T(\mathbb{R}) = \{ f \in \mathcal{C}(\mathbb{R}) \mid f \text{ is periodic with period } T \}$. In general, $\mathcal{C}_T(\mathbb{R})$ might be defined to consist of real valued or of complex valued functions; in this problem we consider only real valued functions.

(a) Let $\mathcal{A} \subset \mathcal{C}_T(\mathbb{R})$ be an algebra (over $\mathbb{R}$) such that $\mathcal{A}$ vanishes at no point and such that if x and y are two points of $\mathbb{R}$ with $x - y$ not an integer multiple of T, then there is an $f \in \mathcal{A}$ with $f(x) \neq f(y)$. Prove: the uniform closure of $\mathcal{A}$ is $\mathcal{C}_T(\mathbb{R})$.

(b) Prove that the set of all finite linear combinations of the functions 1, $\sin nx$, $n \in \mathbb{N}$, and $\cos nx$, $n \in \mathbb{N}$, is dense in $\mathcal{C}_{2\pi}(\mathbb{R})$.

Remarks, hints, and extra questions:

21. The point is that when considering complex-valued functions the hypothesis that the algebra be self-adjoint is really necessary for the Stone-Weierstrass theorem to hold.

22. I suggest that you not only compare with Exercise 6.12, but use it.

23. This is straightforward to check, given the hints. Here is one way that Rudin might have guessed the recursion given here. Finding $|x|$ is the same as solving for a positive root of $f(t) = x^2 - t^2$, and applying Newton's method to this problem leads to the recursion $t_{n+1} = t_n - (x^2 - t_n^2)/2t_n$. This is fairly close to the recursion in the problem, but has two defects: the t_n in the denominator would probably make uniformity difficult to establish (I have not tried) and, even worse, the dependence of t_n on x is not polynomial.

3.A. For (a): the mapping $\phi(x) = e^{ix}$ maps $\mathbb{R}$ onto the unit circle S^1 in the complex plane: $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$. One can use ϕ to set up a bijection of of $\mathcal{C}_T(\mathbb{R})$ with $\mathcal{C}(S^1)$.

*Turn in starred problems (and the extra credit problem, if you do it) Wednesday 2/11.