Assignment 4

Due Wednesday February 18

Exercises:

Chapter 8: 1*, 2*, 3

4.A* (Review of Calculus II) Give, with proof, examples of power series $\sum_{n=0}^{\infty} c_n z^n$, $z \in \mathbb{C}$, with radius of convergence R satisfying $0 < R < \infty$, such that the series

(a) Converges uniformly on the disk $|z| \le R$;

(b) Does not converge for any z with $|z| = R$;

(c) Converge for some z with $|z| = R$ and not for other such z.

4.B* If f and h are real-valued functions defined on $\mathbb{R}$, Riemann integrable on any closed interval, we define the function $f * h$ on $\mathbb{R}$ by

$$(f * h)(x) = \int_{-\infty}^{\infty} f(t)h(x-t)\,dt,$$

whenever this improper Riemann integral exists for all $x \in \mathbb{R}$.

(a) Prove that if $h \ge 0$, $\int_{-\infty}^{\infty} h(x)\,dx$ exists, and f is bounded, then $f * h$ and $h * f$ are defined and equal.

Now let f be continuous and bounded on $\mathbb{R}$, let $h(x) = (2\pi)^{-1/2} e^{-x^2/2}$, and for $n \in \mathbb{N}$ define $h_n(x) = nh(nx)$. You may assume without proof that $\int_{-\infty}^{\infty} h(x)\,dx = 1$ (but you certainly should know how to verify this, even if not with all the rigor of Math 411–412).

(b) Prove that the sequence $(h_n * f)_{n\in\mathbb{N}}$ converges uniformly to f on any interval $[a, b]$. **Hint:** Mimic part of the proof of Theorem 7.26.

(c) Prove that, for each $n \in \mathbb{N}$, $h_n * f$ is differentiable on $\mathbb{R}$. (Thus, every continuous function can be well approximated by differentiable functions.) You may use without proof basic facts from freshman calculus about the exponential function.

(d) **Extra credit:** Prove that, for each $n \in \mathbb{N}$, $h_n * f$ is infinitely differentiable on $\mathbb{R}$.

Remarks and hints: (i) In Rudin 8.1, an induction is obviously in order. *Warning:* you can obtain formulas for all derivatives $f^{(n)}(x)$ at points $x \ne 0$ (you don't really need, and probably don't want, explicit formulas, but you can establish inductively the pattern of these formulas). But just as for f itself, $f^{(n)}(0)$ will have to be given by a separate formula; the general one won't be defined at $x = 0$. *This means that* $f^{n+1)}(0)$ *will always have to be computed directly from the definition of derivative.*

(ii) In 4.B(a), please prove carefully the existence of the improper integrals defining the convolutions, from the definition $\int_{-\infty}^{\infty} g(x)\,dx = \lim_{a,b\to\infty} \int_{-a}^{b} g(x)\,dx$ does. You might find it convenient to formulate and prove a result about for improper intervals over all of $\mathbb{R}$ parallel to the existence part of Exercise 7.12. I am asking you to do this again because many people did not really prove existence in that earlier problem. Please note: we have essentially no theorems available about improper integrals; nothing, for example, like "absolute convergence implies convergence".

*Turn in starred problems Wednesday 2/18.