Assignment 7

Due Wednesday, March 25

Exercises:

Chapter 8: 19*, 30, 31*

7.A* We say that a continuous function $g : [a, b] \to \mathbb{C}$ is *piecewise continuously differentiable* if there exists points $x_0, x_1, \ldots, x_N$, with $a = x_0 < x_1 < \cdots < x_N = b$, such that the restriction of g to each interval $[x_{i-1}, x_i]$ is continuously differentiable. Note that this implies that g has a left derivative at a and a right derivative at b; moreover, g has both left and right derivatives at all other x_i, but that these may be unequal. For example, $|x|$ is piecewise continuously differentiable on $[-1, 1]$.

Now suppose that $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic and continuously differentiable, and that f' is piecewise continuously differentiable on any closed interval. Prove that the Fourier series of f converges uniformly to f. (Hint: integrate by parts in the formula for c_n. An example of such a function is given in Problem 8.14.)

7.B* Following Rudin (Theorem 8.20) we define the beta function by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \qquad x, y > 0.$$

In this problem we obtain the asymptotic behavior of $B(x+1, y+1)$ as x and y become large while keeping a fixed ratio; specifically, for c, d with $0 < c, d < 1$ and $c + d = 1$ we study the behavior of $B(zc+1, zc+1)$ as $z \to \infty$.

(a) We discussed in class a method for obtaining asymptotic behavior, as $x \to \infty$, of certain integrals of the form $\int_a^b \exp[xf(t)]\, dt$. Use this method to study the problem posed above; you should find some "simple" function $F_{c,d}(z)$ such that

$$\lim_{z\to\infty} \frac{B(zc+1, zd+1)}{F_{c,d}(z)} = 1.$$

You do not need to prove rigorously that your answer is correct.

(b) Check your answer in (a) by using Stirling's formula and equation (96) of Rudin.

(c) $B(n+1, m+1)$, n, m integers, is what familiar quantity? Hint: equation (96) again.

Remarks and hints: Except for 8.19, these are rather computational; I hope they don't take too long. HAVE A GOOD BREAK.

8.19. The suggested proof works but is not very helpful in understanding what is going on. Intuitively, the formula holds because the left side is an average of values of f at a large number of points, which may be viewed as selected pseudo-randomly from $[0, 2\pi]$. This is an example of an *ergodic theorem*.

8.30 Before hitting this with Stirling, understand why it is true for c a positive integer (Theorem 8.18(a)).

*Turn in starred problems Wednesday 3/25.