Differential Forms

Our treatment of differential forms departs significantly from that of Rudin [1]; these notes are intended as a *brief* summary of our approach, concentrating on definitions rather than proofs or motivational arguments. Our approach is based to some extent on the very readable little book by Spivak [2].

1. The basic definitions

Throughout we let J denote the interval $[0,1]$ and for $k \geq 1$ write $J^k = J \times \cdots \times J$ (k factors); we also make the convention that $J^0 = \{0\}$. We let E denote an open set in $\mathbb{R}^n$ which is, for most purposes, fixed. We are going to assume that all the functions used in our definitions are infinitely differentiable; this is by no means necessary, but it simplifies the presentation a bit because we never have to ask how many derivatives we really need to assume.

Definition 1: A *singular k-cube* $\boldsymbol{\varphi}$ in E is a $\mathcal{C}^\infty$ mapping $\boldsymbol{\varphi} : J^k \to E$.

Note that, since J^0 consists of a single point, a singular 0-cube $\boldsymbol{\varphi} : J^0 \to E$ picks out a single point of E; we will often think of a 0-cube as the point rather than the map. A singular 1-cube $\boldsymbol{\varphi} : [0,1] \to E$ is just a curve in E.

Definition 2: A k-form ω in E is a function which assigns to every singular k-cube $\boldsymbol{\varphi}$ a number $\int_{\boldsymbol{\varphi}} \omega$, calculated according to the following prescriptions. We write $\mathcal{E}_k(E)$ for the set of all k-forms in E.

(i) $k = 0$: A 0-form may be identified with a $\mathcal{C}^\infty$ function $f : E \to \mathbb{R}$; then if $\boldsymbol{\varphi}$ is a singular 0-cube we define $\int_{\boldsymbol{\varphi}} f = f(\boldsymbol{\varphi}(0))$. Note that if we identify $\boldsymbol{\varphi}$ with the point $\mathbf{a} = \boldsymbol{\varphi}(0) \in E$ as suggested in Definition 1 then $\int_{\boldsymbol{\varphi}} f = \int_{\mathbf{a}} f = f(\mathbf{a})$. For zero forms we usually do not use the integral notation, writing simply $f(\mathbf{a})$.

(ii) $k \geq 1$: If ω is a k-form in E with $k \geq 1$ then there exist $\mathcal{C}^\infty$ functions $a_{i_1 i_2 \ldots i_k} : E \to \mathbb{R}$, indexed by k-tuples $i_1, \ldots, i_k$ with $1 \leq i_j \leq n$, such that for each singular k-cube $\boldsymbol{\varphi}$,

$$\int_{\boldsymbol{\varphi}} \omega = \int_{J^k} \sum_{i_1,\ldots,i_k} a_{i_1 i_2 \ldots i_k}(\boldsymbol{\varphi}(\mathbf{t})) \frac{\partial(x_{i_1},\ldots,x_{i_k})}{\partial(t_1,\ldots,t_k)}\, d\mathbf{t}. \tag{1}$$

Here we have written $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{t})$, $\mathbf{t} \in J^k$, and of course

$$\frac{\partial(x_{i_1},\ldots,x_{i_k})}{\partial(t_1,\ldots,t_k)} = \begin{vmatrix} D_1\varphi_{i_1} & \cdots & D_k\varphi_{i_1} \\ \vdots & \ddots & \vdots \\ D_1\varphi_{i_k} & \cdots & D_k\varphi_{i_k} \end{vmatrix}. \tag{2}$$

The formula (1) is abbreviated as

$$\omega = \sum_{i_1,\ldots,i_k} a_{i_1 i_2 \ldots i_k}(\mathbf{x})\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}. \tag{3}$$

Because the determinant in (2) vanishes when two rows are equal, any coefficient $a_{i_1 i_2 \ldots i_k}$ with $i_p = i_q$ for some $1 \le p < q \le k$ does not contribute to (1) and may be omitted from (3) if we choose; another way to say this is that $dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} = 0$ if there are any repeated indices among $i_1, \ldots, i_k$. An immediate consequence is that *there is no nonzero k-form in E if $k > n$.*

Similarly, because the determinant changes sign when any two rows are interchanged we have

$$dx_{i_{\sigma(1)}} \wedge dx_{i_{\sigma(2)}} \wedge \cdots \wedge dx_{i_{\sigma(k)}} = \operatorname{sgn}(\sigma)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

for any permutation σ of the set $\{1, \ldots, k\}$, with $\operatorname{sgn}(\sigma)$ the sign of the permutation. These facts together mean that (3) can be rewritten in the form

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} b_{i_1 i_2 \ldots i_k}(\mathbf{x})\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}, \tag{4}$$

where

$$b_{i_1 i_2 \ldots i_k} = \sum_{\sigma} \operatorname{sgn}(\sigma)\, a_{i_{\sigma(1)} i_{\sigma(2)} \ldots i_{\sigma(k)}}.$$

Rudin calls (4) the *standard presentation* of the k-form ω. For example, here is a 2-form in $\mathbb{R}^3$, written out in the form (1), and in its standard presentation:

$$\begin{aligned} &a_{11}\, dx \wedge dx + a_{12}\, dx \wedge dy + a_{13}\, dx \wedge dz + a_{21}\, dy \wedge dx \\ &\quad + a_{22}\, dy \wedge dy + a_{23}\, dy \wedge dz + a_{31}\, dz \wedge dx + a_{32}\, dz \wedge dy + a_{33}\, dz \wedge dz \\ &\qquad = (a_{12} - a_{21})\, dx \wedge dy + (a_{13} - a_{31})\, dx \wedge dz + (a_{23} - a_{32})\, dy \wedge dz. \end{aligned}$$

We will sometimes adopt Rudin's notation for (4): $\omega = \sum_I b_I(\mathbf{x})\, dx_I$, in which I denotes an ordered k-tuple, $I = (i_1, \ldots, i_k)$ with $i_1 < \cdots < i_k$, and $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$.

Example 3: Let $\boldsymbol{\varphi}$ be the singular 2-cube in $\mathbb{R}^3$ defined for $a, b > 0$ by

$$\boldsymbol{\varphi}(s,t) = \bigl(a \cos 2\pi s,\ a \sin 2\pi s,\ bt\bigr);$$

this is the lateral surface of a cylinder of radius a and height b. We evaluate the simple example $\int_{\boldsymbol{\varphi}} \omega$ for $\omega = (e^{xz}\, dx \wedge dy + y^2\, dx \wedge dz + x\, dy \wedge dz)$, using

$$\frac{\partial(x,y)}{\partial(s,t)} = 0, \qquad \frac{\partial(x,z)}{\partial(s,t)} = -2\pi ab \sin 2\pi s, \qquad \frac{\partial(y,z)}{\partial(s,t)} = 2\pi ab \cos 2\pi s,$$

to obtain

$$\int_{\boldsymbol{\varphi}} \omega = \int_0^1 \int_0^1 (-2\pi a^3 b \sin^3 2\pi s + 2\pi a^2 b \cos^2 2\pi s)\, ds\, dt = \pi a^2 b.$$

(Computational hint: $\int_0^1 \sin^3 2\pi s\, ds = \int_{-1/2}^{1/2} \sin^3 2\pi s\, ds = 0$ by symmetry; $\int_0^1 \cos^2 2\pi s\, ds = 1/2$ because the average value of $\sin^2$ or $\cos^2$ over one (or one half) period is always 1.)