## 2. Things to do with forms

**2.1 Linear combinations of forms:** If $\omega = \sum_I b_I\,dx_I$ and $\kappa = \sum_I c_I\,dx_I$ are two $k$-forms then the linear combination $r\omega + s\kappa$, where $r, s \in \mathbb{R}$, is defined by the requirement that $\int_{\boldsymbol{\varphi}} (r\omega + s\kappa) = r\int_{\boldsymbol{\varphi}} \omega + s\int_{\boldsymbol{\varphi}} \kappa$ for every $k$-cube $\boldsymbol{\varphi}$. This is equivalent to $r\omega + s\kappa = \sum_I (rb_I + sc_I)\,dx_I$.

**2.2 The wedge product of forms:** If $\omega = \sum_I b_I\,dx_I$ is a $k$-form in $E$ and $\kappa = \sum_{I'} c_{I'}\,dx_{I'}$ an $l$-form in $E$ then their *wedge product* $\omega \wedge \kappa$ is a $(k+l)$-form in $E$ defined by

$$\omega \wedge \kappa = \sum_{I,I'} b_I(\mathbf{x})c_{I'}(\mathbf{x})\,dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{i'_1} \wedge \cdots \wedge dx_{i'_l}. \tag{5}$$

Note that the product $dx_I \wedge\ dx_{I'} = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{i'_1} \wedge \cdots \wedge dx_{i'_l}$ occurring in (5) is already defined, but that (5) is *not* in standard form: $dx_I \wedge dx_{I'}$ will vanish for those terms in which the index sets $I$ and $I'$ are not disjoint, and when it is nonzero the individual $dx_i$ in the product may not occur in increasing order.

**Example 4:** If $\lambda = a(x,y,z)\,dx + b(x,y,z)\,dy + c(x,y,z)\,dz$ and $\rho = \alpha(x,y,z)\,dx + \beta(x,y,z)\,dy + \gamma(x,y,z)\,dz$ are 1-forms in $\mathbb{R}^3$ then

$$\begin{aligned}\lambda \wedge \rho &= a\,\alpha\,dx \wedge dx + a\,\beta\,dx \wedge dy + a\,\gamma\,dx \wedge dz + b\,\alpha\,dy \wedge dx + b\,\beta\,dy \wedge dy \\ &\quad + b\,\gamma\,dy \wedge dz + c\,\alpha\,dz \wedge dx + c\,\beta\,dz \wedge dy + c\,\gamma\,dz \wedge dz \\ &= (a\,\beta - b\,\alpha)\,dx \wedge dy + (a\,\gamma - c\,\alpha)\,dx \wedge dz + (b\,\gamma - c\,\beta)\,dy \wedge dz.\end{aligned}$$

**Proposition 5:** *(i) If If $\omega$ and $\kappa$ are $k$-forms and $\mu$ is an $l$-form then $(r\,\omega + s\,\kappa) \wedge \mu = r\,\omega \wedge \mu + s\,\kappa \wedge \mu$; similarly $\omega \wedge (r\,\kappa + s\,\mu) = r\,\omega \wedge \kappa + s\,\omega \wedge \mu$ if $\omega \in \mathcal{E}_k$ and $\kappa, \mu \in \mathcal{E}_l$.*
*(ii) If $\omega \in \mathcal{E}_k(E)$ and $\kappa \in \mathcal{E}_l(E)$ then $\omega \wedge \kappa = (-1)^{kl}\kappa \wedge \omega$.*

**2.3 Differentiation of forms:** If $\omega$ is a $k$-form in $E$ then its derivative $d\omega$ is a $(k+1)$-form in $E$, that, is, we may think of $d$ as an operator $d : \mathcal{E}_k(E) \to \mathcal{E}_{k+1}(E)$. $d$ is defined on 0-forms by

$$df = \sum_{i=1}^{n} (D_i f)\,dx_i;$$

note that $df$ is indeed a 1-form. For $k \geq 1$, $d$ is defined on the $k$-form $\omega = \sum_I b_I\,dx_I$ by

$$d\omega = \sum_I (db_I) \wedge dx_I \ = \sum_I \sum_{i=1}^{n} (D_i b_I)\,dx_i \wedge dx_I. \tag{6}$$

Again $\sum_I$ runs over ordered $k$-tuples $i_1, \ldots, i_k$. Note that, although we assumed that $\omega$ was written in its standard presentation, $d\omega$ in(6) is not so written; $dx_i \wedge dx_I = dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ may contain repeated indices (in which case it is zero), and the factors $dx_j$ in it will probably be out of standard order.

**Example 6:** Suppose that $p_i$, $1 \le i \le n$, is the 0-form (i.e., function) on $\mathbb{R}^n$ defined by $p_i(\mathbf{x}) = x_i$, that is, the $i^{th}$ coordinate function. Then $dp_i = \sum_{j=1}^n (D_j p_i)(\mathbf{x})\, dx_j = dx_i$. In fact, however, we are making needless distinctions; we might just as well identify $p_i$ with $x_i$, i.e., treat $x_i$ as a function of $\mathbf{x}$ on $\mathbb{R}^n$, and then the conclusion of the above calculation is that $dx_i = dx_i$, where the "$dx_i$" on the left side is the differential of the coordinate function and the "$dx_i$" on the right side is the symbol introduced in the definition of differential forms as part of an abbreviation for the formula (1). The point is that this symbol, which up to this point was purely formal, now has a real meaning in terms of the operator $d$.

**Example 7:** If $\lambda = a\, dx + b\, dy + c\, dz$ is as in Example 4 then (writing $D_1$ as $D_x$, etc.)

$$\begin{aligned} d\lambda &= (D_x a\, dx + D_y a\, dy + D_z a\, dz) \wedge dx + (D_x b\, dx + D_y b\, dy + D_z b\, dz) \wedge dy \\ &\qquad + (D_x c\, dx + D_y c\, dy + D_z c\, dz) \wedge dz \\ &= (D_x b - D_y a)\, dx \wedge dy + (D_x c - D_z a)\, dx \wedge dz + (D_y c - D_z b)\, dy \wedge dz. \end{aligned}$$

**Proposition 8:** *(i) If $\omega$ and $\kappa$ are $k$-forms and $r, s \in \mathbb{R}$ then $d(r\omega + s\kappa) = r\, d\omega + s\, d\kappa$.*
*(ii) $d^2 = 0$, i.e., for any $k$-form $\omega$, $d^2\omega = d(d\omega) = 0$.*
*(iii) If $\omega$ is a $k$-form and $\kappa$ an $l$-form then $d(\omega \wedge \kappa) = (d\,\omega) \wedge \kappa + (-1)^k \omega \wedge (d\kappa)$.*

*Proof sketch:* (i) is trivial and (iii) is easy. For (ii), the key idea is that in calculating $d^2\omega$ one always encounters expressions of the form $D_i(D_j a)\, dx_j \wedge dx_i + D_j(D_i a)\, dx_i \wedge dx_j$, where $a$ is one of the coefficient function in $\omega$. But such an expression is zero, because $D_i D_j a = D_j D_i a$ but $dx_j \wedge dx_i = -dx_i \wedge dx_j$. ■

There are some special relations between forms and vector fields that hold in three dimensions. Suppose then that $n = 3$ and let $\mathbf{v} : E \to \mathbb{R}^3$ be a $\mathcal{C}^\infty$ *vector field* in E; this means that if $(x, y, z) \in E$ then we think of $\mathbf{v}(x, y, z)$ as a little vector attached by its tail to $(x, y, z)$, which tells us the velocity of some fluid at $(x, y, z)$, or the electric field at $(x, y, z)$, etc. If $f \in \mathcal{C}^\infty(E) = \mathcal{E}_0(E)$ then the gradient $\nabla f$ is a vector field.

With the vector field $\mathbf{v}$ there are naturally associated two differential forms in $E$, a 1-form which we denote $\lambda_{\mathbf{v}}$ and a 2-form $\omega_{\mathbf{v}}$, defined by

$$\begin{aligned} \lambda_{\mathbf{v}} &= v_1(x, y, z)\, dx + v_2(x, y, z)\, dy + v_3(x, y, z)\, dz, \\ \omega_{\mathbf{v}} &= v_1(x, y, z)\, dy \wedge dz + v_2(x, y, z)\, dz \wedge dx + v_3(x, y, z)\, dx \wedge dy. \end{aligned}$$

Note that here $\omega_{\mathbf{v}}$ is not quite written in the standard presentation; in dimension 3 it is often convenient to express 2-forms in terms of $dz \wedge dx$ rather than $dx \wedge dz$. Then by calculation as in Example 7 we find that

$$\begin{aligned} df &= \lambda_{\nabla f} \\ d\lambda_{\mathbf{v}} &= \omega_{\nabla \times \mathbf{v}}, \\ d\omega_{\mathbf{v}} &= (\nabla \cdot \mathbf{v})\, dx \wedge dy \wedge dz. \end{aligned}$$

Here $\nabla \times \mathbf{v}$ and $\nabla \cdot \mathbf{v}$ are the usual curl and divergence of the vector field $\mathbf{v}$. Then $d^2 f = 0$ and $d^2 \lambda_{\mathbf{v}} = 0$ express respectively the facts that the curl of a gradient, and the divergence of a curl, are 0.

**2.4 Mapping of forms:** With $E \subset \mathbb{R}^n$ open as usual we suppose that $V$ is an open subset of $\mathbb{R}^m$ and that $T : E \to V$ is $\mathcal{C}^\infty$. We will write $\mathbf{x}$ as a typical point of $E$ and $\mathbf{y}$ as a typical point of $V$, and write $T(\mathbf{x}) = (\mathbf{t}_1(\mathbf{x}), \mathbf{t}_2(x), \ldots, \mathbf{t}_m(\mathbf{x}))$. With every $k$-form $\omega = \sum_I a_I(\mathbf{y})\, dy_I$ in $V$ there is then associated a certain $k$-form in $E$, denoted $\omega_T$:

$$\omega_T = \sum_{0 \le i_1 \le \cdots \le i_k \le 1} a_{i_1 \ldots i_k}(T(\mathbf{x}))\, dt_{i_1} \wedge \cdots \wedge dt_{i_k}. \tag{7}$$

In particular, if $f \in \mathcal{E}_0(V)$ then $f_T = f \circ T$. (This is Rudin's notation; Spivak uses the the more conventional notation $T^*\omega$ for the form defined in (7), so that $T^*$ is a mapping with $T^* : \mathcal{E}_k(V) \to \mathcal{E}_k(E)$.)

**Proposition 9:** *If $E$, $V$, and $T$ are as above, and $\omega$ and $\kappa$ are forms in $V$ then:*
*(i) $(r\omega + s\kappa)_T = r\omega_T + s\kappa_T$ for $\omega, \kappa \in \mathcal{E}_k(V)$, $r, s \in \mathbb{R}$;*
*(ii) $(\omega \wedge \kappa)_T = \omega_T \wedge \kappa_T$;*
*(iii) $d(\omega_T) = (d\omega)_T$;*
*(iv) $\int_{\boldsymbol{\varphi}} \omega_T = \int_{T \circ \boldsymbol{\varphi}} \omega$ if $\omega$ is a $k$-form and $\boldsymbol{\varphi}$ a singular $k$-cube in $E$.*

*Proof sketch:* (i) is immediate and (ii) is very easy. For (iii), consider first $\omega = f \in \mathcal{E}_0(V)$:

$$\begin{aligned} d(f_T) = d(f \circ T) &= \sum_{i=1}^n D_i(f \circ T) dx_i \\ &= \sum_{i,1}^n \sum_{j=1^m} ((D_j f) \circ T) T'_{ji} dx_i = \sum_{j=1^m} ((D_j f) \circ T) dt_j = (df)_T. \end{aligned} \tag{8}$$

Then if $\omega = \sum_I a_I(\mathbf{y})\, dy_I$ is a $k$-form in $V$ with $k \ge 1$, so that with the obvious notation $dt_I = dt_{i_1} \wedge \cdots \wedge dt_{i_k}$ we have $\omega_T = \sum_I (a_I \circ T)\, dt_I = \sum_I (a_I)_T\, dt_I$,

$$\begin{aligned} (d\omega)_T = \left( \sum_I (da_I) \wedge dy_I \right)_T &= \sum_I (da_I)_T \wedge dt_I \\ &= \sum_I d((a_I)_T) \wedge dt_I = d\left( \sum_I (a_I)_T \wedge dt_I \right) = d(\omega_T). \end{aligned} \tag{9}$$

This rather compact derivation shows the power of the formalism: at the third step we used (8); at the fourth we used parts (ii) and (iii) of Proposition 8.

(iv) The case $k = 0$ is trivial; we do the case $k = 1$ and leave $k \ge 2$ as an exercise. If $\lambda = \sum_{j=1}^m a_j(\mathbf{y})\, dy_j$ is a 1-form in $E$ then $\lambda_T = \sum_j (a_j \circ T)(\mathbf{x})\, dt_j$ and for a 1-cube $\boldsymbol{\varphi}$,

$$\begin{aligned} \int_{\boldsymbol{\varphi}} \lambda_T = \int_{\boldsymbol{\varphi}} \sum_{j=1}^m (a_j \circ T)(\mathbf{x})\, dt_j &= \int_{\boldsymbol{\varphi}} \sum_{j=1}^m (a_j \circ T)(\mathbf{x}) \sum_{i=1}^n T'_{ji}(\mathbf{x})\, dx_i \\ &= \int_0^1 \sum_{i,j} a_j(T(\boldsymbol{\varphi}(s))) T_{ji}(\boldsymbol{\varphi}(s)) \varphi'_i(s)\, ds \\ &= \int_0^1 \sum_j a_j(T(\boldsymbol{\varphi}(s)))(T \circ \boldsymbol{\varphi})'_i(s)\, ds = \int_{T \circ \boldsymbol{\varphi}} \lambda. \end{aligned}$$