3. Stokes' Theorem for chains

Before defining chains we need to define certain special singular k-cubes. Let U be an open subset of $\mathbb{R}^k$ with $J^k \subset U$. We will let $\boldsymbol{\psi}(= \boldsymbol{\psi}^k) : J^k \to U$ denote the singular k-cube in U defined by $\boldsymbol{\psi}(\mathbf{s}) = \mathbf{s}$, $\mathbf{s} \in J^k$. For $1 \le i \le k$ and $\alpha = 0, 1$ we let $\boldsymbol{\psi}_{i\alpha}(= \boldsymbol{\psi}^k_{i\alpha}) : J^{k-1} \to U$ denote the singular $(k-1)$-cube in U defined by

$$\boldsymbol{\psi}_{i\alpha}(s_1, \ldots, s_{k-1}) = (s_1, \ldots, s_{i-1}, \alpha, s_i, \ldots, s_{k-1}), \qquad (s_1, \ldots, s_{k-1}) \in J^k.$$

Here $\boldsymbol{\psi}$ should be thought of as providing a way to describe the *cube* J^k, which is a *set*, as a *singular cube*, which by our definition is a *mapping* of J^k into an open set. The set U plays only the minor role of providing the open set into which $\boldsymbol{\psi}$ maps. Similarly $\boldsymbol{\psi}_{i\alpha}$ provides a description of one of the faces of J^k—that is, of a subset of J^k—as a singular $(k-1)$-cube. $\boldsymbol{\psi}_{i1}$ and $\boldsymbol{\psi}_{i0}$ describe the two faces of J^k which are perpendicular to the i^{th} coordinate axis; one might call them the "top" and "bottom" faces, respectively.

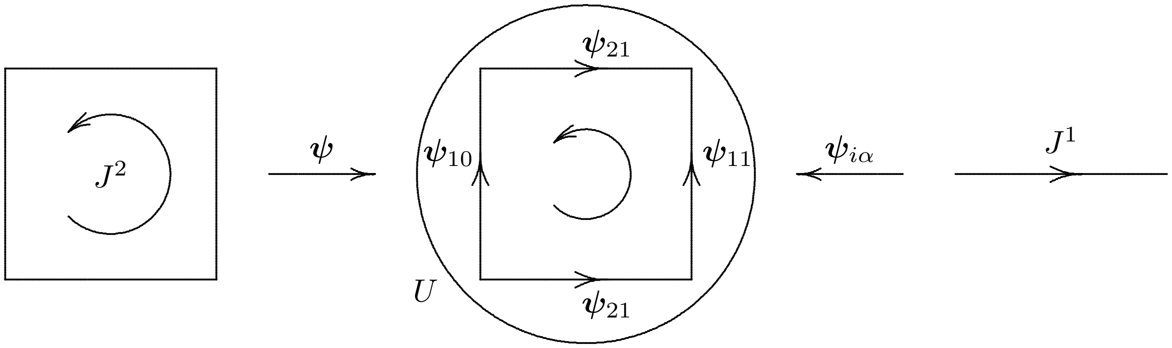

Figure 1. The singular 2-cube $\boldsymbol{\psi} = \boldsymbol{\psi}^2$ and 1-cubes $\boldsymbol{\psi}_{i\alpha} = \boldsymbol{\psi}^2_{i\alpha}$.

The figure shows what is going on for $k = 2$. We have added one element: we are thinking of our cubes as *oriented*, as shown by the arrows in the picture. The singular cubes $\boldsymbol{\psi}$ and $\boldsymbol{\psi}_{i\alpha}$ inherit their orientation from orientation on the source cubes J^2 and J^1; if we changed these maps, for example by defining $\hat{\boldsymbol{\psi}}(s_1, s_2) = (s_2, s_1)$ or $\hat{\boldsymbol{\psi}}_{1,0}(s) = (0, 1-s)$, then the orientation of the singular cubes would change.

Remark 10: We will use the singular cube $\boldsymbol{\psi}$ in the following context. Suppose that $\boldsymbol{\varphi}$ is a singular k-cube in some open set $E \subset \mathbb{R}^n$, that is, a $\mathcal{C}^\infty$ map $\boldsymbol{\varphi} : J^k \to E$, and that ω is a k-form in E. Then by definition (of a C^∞ map on a compact set) there is some open set $U \subset \mathbb{R}^k$ with $J^k \subset U$ and a map $\boldsymbol{\varphi} : U \to E$ whose restriction to J^k is the original given $\boldsymbol{\varphi}$ (we will not distinguish between these two versions of $\boldsymbol{\varphi}$). Now $\omega_{\boldsymbol{\varphi}}$ is a k-form in U, and since $\boldsymbol{\varphi} = \boldsymbol{\varphi} \circ \boldsymbol{\psi}$ we have from Proposition 9 (iv) that

$$\int_{\boldsymbol{\varphi}} \omega = \int_{\boldsymbol{\varphi} \circ \boldsymbol{\psi}} \omega = \int_{\boldsymbol{\psi}} \omega_{\boldsymbol{\varphi}}. \tag{10}$$

The last formula in (10) is essentially our original formula (1). For if we write $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{s}) = (\varphi_1(\mathbf{s}), \ldots, \varphi_n(\mathbf{s}))$ then

$$d\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} = \frac{\partial(x_{i_1}, \ldots, x_{i_k})}{\partial(s_1, \ldots, s_k)}\, ds_1 \wedge \cdots \wedge ds_k \tag{11}$$

(Exercise: verify (11)) and so if $\omega = \sum_I a_I(\mathbf{x})\,dx_I$,

$$\begin{aligned}\int_{\boldsymbol{\varphi}} \omega = \int_{\boldsymbol{\psi}} \omega_{\boldsymbol{\varphi}} &= \int_{\boldsymbol{\psi}} \sum_I (a_I \circ \phi)\, d\varphi_{i_1} \wedge \cdots \wedge d\varphi_{i_k} \\ &= \int_{\boldsymbol{\psi}} \sum_I a_I(\boldsymbol{\varphi}(\mathbf{s})) \frac{\partial(x_{i_1}, \ldots, x_{i_k})}{\partial(s_1, \ldots, s_k)}\, ds_1 \wedge \cdots \wedge ds_k,\end{aligned} \tag{12}$$

and this is almost exactly (1). There remains a slight but important difference: (12) is still an integral of a form over a singular cube, and in (1) this has been reduced to the integral of a function over a geometric cube.

Definition 11: (a) Suppose that $E \subset \mathbb{R}^n$ is open. A *singular k-chain* γ in E is a formal finite sum of the form

$$\gamma = m_1\boldsymbol{\varphi}_1 + \cdots + m_r\boldsymbol{\varphi}_r = \sum_{i=1}^{r} m_i\boldsymbol{\varphi}_i, \tag{13}$$

where each $\boldsymbol{\varphi}_i$ is a singular k-cube in E and $m_1, \ldots, m_r$ are integers. Chains may be added and subtracted in the obvious way, for example,

$$(3\boldsymbol{\varphi}_1 - 2\boldsymbol{\varphi}_2) - 2(2\boldsymbol{\varphi}_1 - 3\boldsymbol{\varphi}_3) = -\boldsymbol{\varphi}_1 - 2\boldsymbol{\varphi}_2 + 6\boldsymbol{\varphi}_3.$$

Moreover, if ω is a k form in E and γ is the chain given in (13) then $\int_\gamma \omega$ is defined to be $\sum_{i=1}^r m_i \int_{\boldsymbol{\varphi}_i} \omega$.

(b) The above is a formal, mathematically imprecise definition, but you should not have trouble working with it. Spivak suggests the following precise definition (which we will not use further): if $\mathcal{S}$ is the set of all singular k-cubes in E then a k-chain is a map $\gamma : \mathcal{S} \to \mathbb{Z}$ such that $\gamma(\boldsymbol{\varphi})$ is non-zero for only finitely many $\boldsymbol{\varphi} \in \mathcal{S}$. Chains are then added as functions usually are: $(\gamma_1 + \gamma_2)(\boldsymbol{\varphi}) = \gamma_1(\boldsymbol{\varphi}) + \gamma_2(\boldsymbol{\varphi})$. One recovers the notation of (a) if one writes $\boldsymbol{\varphi}$ not only for the k-cube $\boldsymbol{\varphi}$ but also for the chain which takes value 1 on this cube and 0 on all others.

Definition 12: The *boundary $\partial\boldsymbol{\varphi}$ of a singular k-cube $\boldsymbol{\varphi}$* in E is a $(k-1)$-chain in E. For $k = 1$, with $\boldsymbol{\varphi} : J \to E$,

$$\partial\boldsymbol{\varphi} = \boldsymbol{\varphi}(1) - \boldsymbol{\varphi}(0). \tag{14}$$

Here on the right side we are using the notation that a singular 0-cube is just a point of E. For $k \geq 2$, with $\boldsymbol{\varphi} : J^k \to E$,

$$\partial\boldsymbol{\varphi} = \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} \boldsymbol{\varphi} \circ \boldsymbol{\psi}^k_{i\alpha}. \tag{15}$$

The *boundary of a k-chain* $\gamma = \sum_{i=1}^r m_i\boldsymbol{\varphi}_i$ is the $(k-1)$-chain $\partial\gamma = \sum_{i=1}^r m_i\partial\boldsymbol{\varphi}_i$.

From (15) one finds that

$$\begin{aligned}\partial\boldsymbol{\psi}^2 &= \boldsymbol{\psi}\circ\boldsymbol{\psi}^2_{20} + \boldsymbol{\psi}\circ\boldsymbol{\psi}^2_{10} - \boldsymbol{\psi}\circ\boldsymbol{\psi}^2_{10} - \boldsymbol{\psi}\circ\boldsymbol{\psi}^2_{10}\\ &= \boldsymbol{\psi}^2_{20} + \boldsymbol{\psi}^2_{10} - \boldsymbol{\psi}^2_{10} - \boldsymbol{\psi}^2_{10}\end{aligned}$$

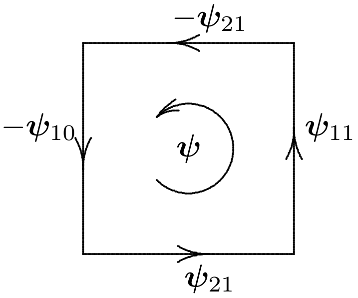

Figure 2. Boundary of $\boldsymbol{\psi}^2$

Working from Figure 1 we see why these signs are appropriate: this boundary may be represented geometrically as in Figure 2. This same geometric interpretation of boundaries in terms of orientation carries over 3-cells, 4-cells, etc. although it is harder to picture. For example, the formula

$$\partial\boldsymbol{\psi}^3 = -\boldsymbol{\psi}^2_{10} + \boldsymbol{\psi}^2_{11} + \boldsymbol{\psi}^2_{20} - \boldsymbol{\psi}^2_{21} - \boldsymbol{\psi}^2_{30} + \boldsymbol{\psi}^2_{31} \tag{16}$$

can be interpreted as follows: each map $\psi^2_{i\alpha}$ provides, by carrying forward the natural orientation of J^2 (see Figure 1), an orientation for the (i,α)-face of the cube J^3; a minus sign reverses this orientation. The resulting orientation of this face corresponds, via the right-hand rule (curl the fingers of your right hand in the direction of the arrow on the face, and look where your thumb points), to an outward normal vector on the face.

Now we can prove Stokes' Theorem for chains.

Theorem 13: *If $E\subset\mathbb{R}^n$ is open, ω is a $(k-1)$-form in E, and γ is a k-chain in E, then*

$$\int_\gamma d\omega = \int_{\partial\gamma}\omega. \tag{17}$$

Proof: We will prove that for $\boldsymbol{\varphi}$ a singular k-cube in E,

$$\int_{\boldsymbol{\varphi}} d\omega = \int_{\partial\boldsymbol{\varphi}}\omega. \tag{18}$$

Then (17) is immediate: for $\gamma=\sum_{i=1}^r m_i\boldsymbol{\varphi}_i$,

$$\int_\gamma d\omega = \sum_i m_i\int_{\boldsymbol{\varphi}_i} d\omega = \sum_i m_i\int_{\partial\boldsymbol{\varphi}_i}\omega = \int_{\partial\gamma}\omega.$$

The case $k=1$ of (18) is easy: $\boldsymbol{\varphi}$ is a 1-cube, $\omega=f$ is a 0-form (a function), $df=\sum_{i=1}^n D_if\,dx_i$, and

$$\int_{\boldsymbol{\varphi}} d\omega = \int_0^1\sum_{i=1}^n (D_if)(\boldsymbol{\varphi}(t))\varphi_i'(t)\,dt = \int_0^1\frac{d}{dt}[f(\boldsymbol{\varphi}(t))]\,dt = f(\boldsymbol{\varphi}(1)) - f(\boldsymbol{\varphi}(0)) = \int_{\partial\boldsymbol{\varphi}} f.$$

To prove (18) for $k\ge 2$ we first consider a special case: $\boldsymbol{\varphi}=\boldsymbol{\psi}(=\boldsymbol{\psi}^k)$, the k-cube in $U\supset J^k$ defined above (that is, we take $E=U\subset\mathbb{R}^k$). Then ω will be a $(k-1)$-form in U, and so must have the form

$$\omega = \sum_{i=1}^k a_i(\mathbf{x})\,dx_1\wedge\cdots dx_{i-1}\wedge dx_{i+1}\wedge\cdots\wedge dx_k. \tag{19}$$

It suffices to consider each term in (19) separately, so we fix i, $1 \le i \le k$, and take

$$\omega = a(\mathbf{x})\, dx_1 \wedge \cdots dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_k.$$

Then

$$\begin{aligned} d\omega &= \sum_{j=1}^{n} (D_j a)(\mathbf{x})\, dx_j \wedge dx_1 \wedge \cdots dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_k \\ &= (-1)^{i-1} (D_i a)(\mathbf{x})\, dx_1 \wedge \cdots \wedge dx_k, \end{aligned}$$

and so, remembering that $\boldsymbol{\psi}(\mathbf{s}) = \mathbf{s}$ for $\mathbf{s} \in J^k$, we have from (1)

$$\begin{aligned} \int_{\boldsymbol{\psi}} d\omega &= (-1)^{i-1} \int_{J^k} (D_i a)(\mathbf{s})\, ds_1 \cdots ds_k \\ &= (-1)^{i-1} \int_0^1 \cdots \int_0^1 \left[\int_0^1 (D_i a)(\mathbf{s})\, ds_i \right] ds_1 \cdots ds_{i-1}\, ds_{i+1} \cdots ds_k \\ &= (-1)^{i-1} \int_0^1 \cdots \int_0^1 \big[a(s_1, \ldots, s_{i-1}, 1, s_{i+1}, \ldots, s_k) \\ &\qquad - a(s_1, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, s_k) \big]\, ds_1 \cdots ds_{i-1}\, ds_{i+1} \cdots ds_k, \end{aligned} \tag{20}$$

where we have used the fact that we can do these integrals in any order we like—in particular, do the i^{th} integral first. On the other hand, from (15) and $\boldsymbol{\psi} \circ \boldsymbol{\psi}_{j\alpha} = \boldsymbol{\psi}_{j\alpha}$,

$$\begin{aligned} \int_{\partial \boldsymbol{\psi}} \omega &= \sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} \int_{\boldsymbol{\psi}_{j\alpha}} \omega \\ &= (-1)^{i+1} \int_{\boldsymbol{\psi}_{i1}} \omega + (-1)^i \int_{\boldsymbol{\psi}_{i0}} \omega \\ &= (-1)^{i+1} \int_0^1 \cdots \int_0^1 a(s_1, \ldots, s_{i-1}, 1, s_i, \ldots, s_{k-1})\, ds_1 \cdots ds_{k-1} \\ &\qquad + (-1)^i \int_0^1 \cdots \int_0^1 a(s_1, \ldots, s_{i-1}, 0, s_i, \ldots, s_{k-1})\, ds_1 \cdots ds_{k-1}, \end{aligned} \tag{21}$$

which agrees with (20) (the change of integration variables $(s_{i+1}, \ldots, s_k) \to (s_i, \ldots, s_{k-1})$ of course does not affect the value of the integral). In the second equality in (21) we have used the fact that if $j \ne i$ then

$$\int_{\boldsymbol{\psi}_{j\alpha}} \omega = \int_{\boldsymbol{\psi}_{j\alpha}} a(\mathbf{x})\, dx_1 \wedge \cdots dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_k = 0. \tag{22}$$

Formally this is true because because x_j is constant on the image of $\psi_{j\alpha}$ and therefore $dx_j = 0$ there. More precisely, evaluation of (22) via (1) involves the Jacobian

$$\frac{\partial(x_1, \ldots, x_{i-1}, x_{i+1} \ldots, x_k)}{\partial(s_1, \ldots s_{k-1})} \tag{23}$$

for $\mathbf{x} = \boldsymbol{\psi}_{j\alpha}(\mathbf{s})$; (23) is the determinant of a certain matrix, one row of which,

$$\bigl[D_1(\boldsymbol{\psi}_{j\alpha}), \ldots, D_{k-1}(\boldsymbol{\psi}_{j\alpha})\bigr],$$

is identically zero since $(\boldsymbol{\psi}_{j\alpha})_j = \alpha$ is constant.

To complete the proof we must show that (18) holds for a general k-cube $\boldsymbol{\varphi}$ and $(k-1)$-form ω in $E \subset \mathbb{R}^n$. But this is now easy:

$$\begin{aligned} \int_{\boldsymbol{\varphi}} d\omega = \int_{\boldsymbol{\psi}} (d\omega)_{\boldsymbol{\varphi}} = \int_{\boldsymbol{\psi}} d(\omega_{\boldsymbol{\varphi}}) = \int_{\partial\boldsymbol{\psi}} \omega_{\boldsymbol{\varphi}} &= \sum_{j,\alpha} (-1)^{j+\alpha} \int_{\boldsymbol{\psi}_{j,\alpha}} \omega_{\boldsymbol{\varphi}} \\ &= \sum_{j,\alpha} (-1)^{j+\alpha} \int_{\boldsymbol{\varphi}\circ\boldsymbol{\psi}_{j,\alpha}} \omega = \int_{\partial\boldsymbol{\varphi}} \omega, \end{aligned}$$

where we have used (10), Proposition 9 (iii), the special case above, the definition of boundary (twice), and Proposition 9 (iv). ■