Application of the Ascoli-Arzela Theorem to an Initial Value Problem

In these notes we consider the problem of solving an ordinary differential equation for an unknown function $x(t)$; one frequently thinks of t as time and x as some quantity of physical interest. We are not interested in constructing an explicit solution but rather in proving the existence of at least one solution. It would be nice to know that the solution is unique, but this may not be true; see Remark 3 below.

Let U be an open subset of $\mathbb{R}^2$, let $f : U \to \mathbb{R}$ be continuous, and let (t^*, x^*) be a point of U. We want to prove a local existence theorem for the corresponding *initial value problem*, written formally as

$$x'(t) = f(t, x(t)), \tag{1}$$

$$x(t^*) = x^*. \tag{2}$$

That is, we would like to prove:

Theorem 1: *There exists a $\tau > 0$ and a differentiable function $\xi : (t^* - \tau, t^* + \tau) \to \mathbb{R}$ such that $\xi(t^*) = x^*$ and for all $t \in (t^* - \tau, t^* + \tau)$, $(t, \xi(t)) \in U$ and $\xi'(t) = f(t, \xi(t))$.*

Remark 2: (a) Here is an example of the sort of problem we are discussing: solve

$$\frac{dx}{dt} = \frac{xe^t - t}{x^2 + t^2} \qquad \text{with} \qquad x(1) = 1.$$

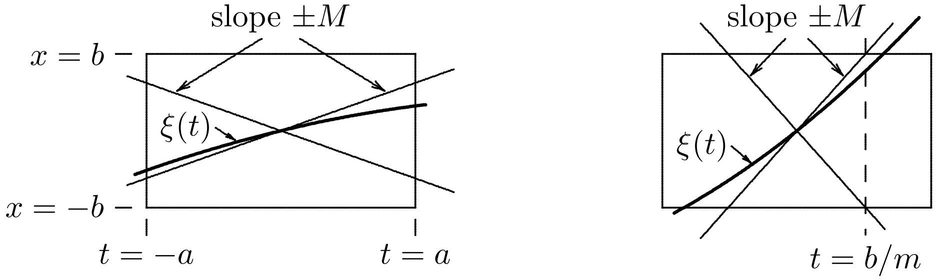
In this example it would be natural to take $U = \{(t, x) \in \mathbb{R}^2 \mid (x, t) \neq (0, 0)\}$.

(b) The theorem is a *local* existence result because we ask only for a solution on some small interval near the initial time. Questions of *global* existence—does the solution exist for all time? if not, to what maximal time interval can the solution be extended?—are more difficult, and to answer these one would typically need more knowledge of the function f.

(c) Realistic problems usually involve more than one unknown function, so that one must solve a differential equation $\vec{x}\,' = \vec{f}(t, \vec{x})$ for a vector $\vec{x}(t) = (x_1(t), \dots, x_n(t))$ of unknown functions; now $U \subset \mathbb{R}^{n+1}$ and $\vec{f} : U \to \mathbb{R}^n$. The proof of Theorem 1 given below extends almost without change to this situation.

In proving the theorem we will, without loss of generality, suppose that $t^* = x^* = 0$. Since U is open and $(0, 0) \in U$ we may find a (closed) rectangle $R = [-a, a] \times [-b, b]$ with $a, b > 0$ and $R \subset U$. Let $M = \max\{|f(t, x)| \mid (t, x) \in R\}$. We will prove the theorem with $\tau = \min\{a, b/M\}$.

We restrict τ in this way—that is, require that $\tau \leq a$ and $\tau \leq b/M$—because we want our solution $\xi(t)$ to be such that $f(t, \xi(t))$ lies in R, the set on which we have the upper bound on $|f| \leq M$. Note that $|\xi'(t)| = |f(t, \xi(t))| \leq M$, so that $\xi(t)$ must satisfy $|\xi(t) - \xi(0)| = |\xi(t)| \leq M|t|$, that is, our solution must lie between the two straight lines which pass through the origin and have slopes $\pm M$. Now certainly we require $\tau \leq a$; this guarantees that $|t| \leq a$, i.e., that our solution cannot leave the rectangle through the end where $|t| = a$. This is all that is required when $a \leq b/M$ (see Figure 1, Case(a)). On the other hand, when $b/M < a$, the further restriction $\tau \leq b/M$ is needed to guarantee that for $|t| \leq \tau$, $|\xi(t)| \leq b$, i.e., that the solution does not leave the rectangle through the top or bottom, where $|x| = b$ (see Figure 1, Case(b))).



Case(a): $\tau = a \leq b/M$ Case (b): $\tau = b/M < a$

Figure 1

Proof of Theorem 1: We will define the function ξ on the interval $[0, \tau]$; the definition on $[-\tau, 0]$ is similar. To do so we will construct a sequence ξ_n of approximate solutions, all defined on $[0, \tau]$, obtain a convergent subsequence ξ_{n_j} by the Ascoli-Arzela Theorem, and then show that the limit $\xi = \lim_{j\to\infty} \xi_{n_j}$ is the desired solution.

Take $n \in \mathbb{N}$. The approximate solution ξ_n is defined by *Euler's method.* It is convenient to define at the same time an auxiliary function $\phi_n : [0, \tau] \to \mathbb{R}$ which essentially plays the role of the derivative of ξ_n. We introduce the set of mesh points $\{t_0, t_1, \ldots, t_n\} \subset [0, \tau]$ with $t_k = k\tau/n$, so that $t_0 = 0$, $t_n = \tau$, and the points of the partition are equally spaced. Now define

$$\phi_n(0) = f(0, 0) \qquad \text{and} \qquad \xi_n(0) = 0. \tag{3}$$

Here the definition of $\phi_n(0)$ is motivated by the role of ϕ_n mentioned above and by (1) and (2), which imply that $\xi'(0) = f(0, 0)$; the second definition implies that each approximate solution satisfies the initial condition (2). Now suppose inductively that

(i) ξ_n and ϕ_n have been defined on the interval $[0, t_k]$;
(ii) $|\phi_n| \leq M$ on $[0, t_k]$;
(iii) ϕ_n is Riemann integrable on $[0, t_k]$;
(iv) $\xi_n(t) = \int_0^t \phi_n(s)\, ds$ for all $t \in [0, t_k]$; and
(v) $(t, \xi_n(t)) \in R$ for all $t \in [0, t_k]$.

These requirements in the case $k = 0$ are trivially satisfied by (3). For the induction step, define for $t \in (t_k, t_{k+1}]$,

$$\phi_n(t) = f(t_k, \xi_n(t_k)) \qquad \text{and} \qquad \xi_n(t) = \xi_n(t_k) + f(t_k, \xi_n(t_k))(t - t_k). \tag{4}$$

Note that $f(t_k, \xi_n(t_k))$ is well defined by (v). Verification of the induction hypotheses is straightforward: (i) is true trivially, (ii) follows from the bound $|f| \leq M$ on R, (iii) holds because ϕ_n is continuous except at the points t_k, (iv) follows from (4) and the induction hypothesis, and (v) follows from the fact that $t_k \leq \tau \leq a$ and from (ii) and (iv), since $|\xi_n(t)| \leq \int_0^t |\phi_n(s)|\, ds \leq Mt \leq b$. Completing the induction through $k = n - 1$, we arrive at a definition of the ξ_n and ϕ_n on all of $[0, t_n] = [0, \tau]$.

Now if $0 \le t < t' \le \tau$, (iv) and (ii) above imply that

$$|\xi_n(t') - \xi_n(t)| = \left| \int_t^{t'} \phi_n(s)\, ds \right| \le \int_t^{t'} |\phi_n(s)|\, ds \le M(t'-t). \tag{5}$$

This means that the sequence (ξ_n) is equicontinuous on $[0,\tau]$, since given $\epsilon > 0$ we see that if $|t'-t| < \epsilon/M$ then $|\xi_n(t') - \xi_n(t)| < \epsilon$ for all n. Thus, by the Ascoli-Arzela therom, (ξ_n) contains a uniformly convergent subsequence $(\xi_{n_j})_{j\in\mathbb{N}}$; let ξ denote the limit of this subsequence. We emphasize that $\lim_{j\to\infty} \xi_{n_j} = \xi$ uniformly on $[0,\tau]$. It will follow from the next claim that ξ is the solution that we seek.

Claim: *The sequence* (ϕ_{n_j}) *converges uniformly on* $[0,\tau]$, *with* $\lim_{j\to\infty} \phi_{n_j}(t) = f(t, \xi(t))$.

Suppose that this claim is established. Then by replacing n by n_j in (iv) above, and taking the $j \to \infty$ limit of the resulting equation, and using Theorem 7.16, we find that

$$\xi(t) = \lim_{j\to\infty} \xi_{n_j}(t) = \lim_{j\to\infty} \int_0^t \phi_{n_j}(s)\, ds = \int_0^t f(s, \xi(s))\, ds. \tag{6}$$

Equation (6) shows that ξ satisfies the correct initial condition $\xi(0) = 0$ and, by Theorem 6.20 (one version of the Fundamental Theorem of Calculus) that $\xi'(t) = f(t, \xi(t))$. Thus ξ is the desired solution.

We now prove the claim. Suppose we are given $\epsilon > 0$. Since f is continuous on the compact set R it is uniformly continuous there, so that there exists an $\eta > 0$ such that if $(x,t), (x',t') \in R$ and $|t-t'| < \eta$, $|x-x'| < \eta$ then $|f(t,x) - f(t',x')| < \epsilon$. By equicontinuity of (ξ_n) there exists a $\delta > 0$ such that $|t-t'| < \delta$ implies $|\xi_n(t) - \xi_n(t')| < \eta/2$ for all n (we may take $\delta = \eta/(2M)$, by (5)). Finally, there exists an $N \in \mathbb{N}$ so large that $\tau/N < \min\{\delta, \eta\}$ and, using uniform convergence of (ξ_n), that if $j \ge N$, $|\xi_{n_j}(t) - \xi(t)| < \eta/2$ for all $t \in [0,\tau]$.

Now take $j \ge N$ (which guarantees that $n_j \ge N$) and consider the difference

$$|\phi_{n_j}(t) - f(t, \xi(t))| \tag{7}$$

for some $t \in [0,\tau]$. If $t = 0$ then (7) vanishes by our definition (3) of $\phi_{n_j}(0)$. Otherwise, we again write $t_0, t_1, \ldots t_{n_j}$ for the subdivision points of $[0,\tau]$ used to define ϕ_{n_j} and ξ_{n_j}; then $t \in (t_k, t_k + 1]$ for some k, $0 \le k \le n_j - 1$, and $\phi_{n_j}(t)$ and $\xi_{n_j}(t)$ are defined by (4). Now

$$t - t_k \le \tau/n_j \le \tau/N < \eta. \tag{8}$$

Moreover, $t - t_k \le \tau/n_j \le \tau/N < \delta$ and so

$$|\xi_{n_j}(t_k) - \xi(t)| \le |\xi_{n_j}(t_k) - \xi_{n_j}(t)| + |\xi_{n_j}(t) - \xi(t)| < \frac{\eta}{2} + \frac{\eta}{2} = \eta. \tag{9}$$

But (8) and (9) imply that

$$|\phi_{n_j}(t) - f(t, \xi(t))| = |f(t_k, \xi_{n_j}(t_k) - f(t, \xi(t))| < \epsilon.$$

This is uniform convergence of (ϕ_{n_j}) to the limiting function $f(t, \xi(t))$. ■

Remark 3: (a) The Ascoli-Arzela Theorem, or something like it, is really needed here; one can construct rather complicated examples in which the original sequence (ξ_n) obtained from Euler's method does not converge.

(b) The solution of the initial value problem (1)–(2) may not be unique. For example, consider the problem $x' = x^{1/3}$, $x(0) = 0$. Obviously, one solution is the identically zero function: $x(t) = 0$ for all t. Separation of variables, on the other hand, leads formally to $x(t) = (2t/3)^{3/2}$; this is not defined for $t < 0$ but leads to solutions defined for all t, for example,

$$x(t) = \begin{cases} (2t/3)^{3/2}, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0, \end{cases} \qquad \text{or} \qquad x(t) = (2/3)^{3/2} t|t|^{1/2}.$$

(c) Theorem 1 is often discussed with an added hypothesis: one supposes that, in addition to being continuous, f satisfies a *Lipschitz condition* in x. This means that there is a constant $C \geq 0$ such that, for any points (x_1, t) and (x_2, t) in U,

$$|f(t, x_1) - f(t, x_2)| \leq C|x_1 - x_2|.$$

C is called a *Lipschitz constant.* Under this hypothesis the solution of the initial value problem (1)–(2) is unique, and there is a proof of the theorem which is somewhat simpler than the one we have given and does not require the use of the Ascoli-Arzela Theorem.

(d) The various ideas above are closely connected. When the initial value problem has a unique solution, e.g., when f satisfies a Lipschitz condition, the sequence (ξ_n) does converge (to that solution). Note that, as expected, the function $f(t, x) = x^{1/3}$ considered in (b) does not satisfy a Lipschitz condition.