SOME SOLUTIONS FOR ASSIGNMENT 1

7.4: We claim that the series: (i) converges absolutely for all x except

$$x = 0 \qquad \text{and} \qquad x = -\frac{1}{n^2}, \quad n \in \mathbb{N}; \tag{1.1}$$

(ii) converges absolutely and uniformly on $(-\infty, -1)$, on $(-1/n^2, -1/(n+1)^2)$ for any $n \in \mathbb{N}$, on $[a, \infty)$ for any $a > 0$, and (hence) on any subinterval of one of these intervals; and

(iii) fails to converge uniformly on any interval containing one of the points (1.1) or containing an interval of the form $(0, a)$, $a > 0$.

Moreover, the sum f is: (iv) continuous, but not bounded, on the complement of the set of points (1.1).

Proof of the claim: (i) Since the nth term of the series is undefined at $x = -1/n^2$, and each term has value 1 at $x = 0$, the series certainly does not converge at any of the points (1.1). It will follow from (ii) that the series converges absolutely at every other point.

(ii) If $a > 0$ then for $x \geq a$, $0 < 1/(1 + n^2 x) < 1/(an^2)$, so the series converges uniformly on $[a, \infty)$ by the Weierstrass M-test. To study the series for negative x we fix $k \geq 2$, $k \in \mathbb{N}$, and consider the terms of the series with $n > k$ for $x \leq -1/k^2$; writing $n = k + m$, $m \in \mathbb{N}$, we then have

$$xn^2 \leq -\left(\frac{n}{k}\right)^2 = -\left(\frac{k+m}{k}\right)^2 < -\left(1 + \left(\frac{m}{k}\right)^2\right),$$

so that $|1 + n^2 x| = |n^2 x| - 1 > (m/k)^2$, from which it follows from the M-test that the series $\sum_{n=k+1}^{\infty} 1/(1 + n^2 x)$ converges uniformly and absolutely on $(-\infty, -1/k^2]$. This shows in turn that the original series converges uniformly and absolutely on $(-\infty, -1)$ and on $(-1/(n-1)^2, -1/n^2)$ for any n with $1 < n \leq k$, since the two series differ by a finite number of terms which are well defined on these intervals.

(iii,iv) The first part of (iii) follows immediately from (i), so we need only show that the series does not converge uniformly on $(0, a)$. Uniform convergence implies that the partial sums of the series are uniformly Cauchy, and hence that the sequence of terms must converge uniformly to 0. But if $0 < x < 1/n^2$ then $1/(1 + xn^2) > 1/2$, so the convergence is not uniform on $(0, a)$, verifying (iii). This estimate also shows that if $0 < x < 1/n^2$ then $f(x) \geq \sum_{k=1}^{n} 1/(1 + xn^2) \geq n/2$, so that f is not bounded. Finally, since each point at which f converges is by (i) and (ii) contained in some open interval on which the convergence is uniform, f is continuous where it is defined. ■

7.8: Uniform convergence is immediate from the M-test, since $|c_n I(x_n - x)| \leq |c_n|$. We claim that f has a left limit $f(x-)$ at each $x \in (a, b]$ and a right limit $f(x+)$ at $x \in [a, b)$; moreover, that

$$f(x-) = f(x); \qquad f(x+) = \begin{cases} f(x) + c_n, & \text{if } x = x_n \text{ for some } n \in \mathbb{N}, \\ f(x), & \text{otherwise.} \end{cases} \tag{1.2}$$

Continuity at points not in $\{x_n \mid n \in \mathbb{N}\}$ and at x_n iff $c_n = 0$ follows immediately.

To verify (1.2) we note first that if $a \leq y < z \leq b$ then

$$f(z) - f(y) = \sum_{\{k \mid y \leq x_k < z\}} c_k. \tag{1.3}$$

Now fix $x \in [a,b]$. Given $\epsilon > 0$ we may choose $N \in \mathbb{N}$ such that $\sum_{k \geq N} |c_k| < \epsilon$, and then $\delta > 0$ such that if $0 < |x_k - x| < \delta$ then $k \geq N$. Then if $x - \delta < y < x$, (1.3) yields

$$|f(x) - f(y)| \leq \sum_{\{k | y \leq x_k < x\}} |c_k| \leq \sum_{k \geq N} |c_k| < \epsilon. \tag{1.4}$$

This establishes the first equality in (1.2). For the second equality, let $c = 0$ if $x \notin \{x_n \mid n \in \mathbb{N}\}$ and $c = c_n$ if $x = x_n$; then if $x < y < x + \delta$ we have from (1.3) that

$$f(y) - f(x) = c + \sum_{\{k | y < x_k < z\}} c_k$$

and an estimate as in (1.4) shows that $|f(y) - (f(x) + c)| < \epsilon$. ■

7.9: We first prove the proposition of the exercise. If $f_n \to f$ uniformly, each f_n and hence f is continuous, and $x_n \to x$, then given $\epsilon > 0$ there is an N_1 with $|f_n(z) - f(z)| < \epsilon/2$ for all $n \geq N_1$ and $z \in E$, there is a $\delta > 0$ such that $|f(z) - f(x)| < \epsilon/2$ for all $z \in E$ with $d(z,x) < \delta$, and there is an N_2 with $d(x_n, x) < \delta$ for $n \geq N_2$. Then for $n \geq \max\{N_1, N_2\}$,

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so $f_n(x_n) \to f(x)$.

We now consider the given converse:

Proposition: If $(f_n)_{n \in \mathbb{N}}$ is a sequence of continuous functions defined on a compact metric space E, f is a function defined on E, and $f_n(x_n) \to f(x)$ for every $x \in E$ and every sequence $(x_n)_{n \in \mathbb{N}}$ of points in E with $x_n \to x$, then (f_n) converges uniformly to f.

Proof: We first observe that (f_n) converges to f pointwise. For if $x \in E$ the sequence (x_n) with $x_n = x$ for all n converges to x and hence $f_n(x_n) \to f(x)$, i.e., $f_n(x) \to f(x)$.

Before proceeding it is convenient to observe that if $y_k \to y$ and (f_{n_k}) is a subsequence of (f_n) then $f_{n_k}(y_k) \to y$. For we may define a sequence (x_n) by setting $x_n = y_k$ if $n = n_k$, $x_n = y$ otherwise; the sequence (x_n) converges to y and hence $f_n(x_n) \to f(y)$; thus also the subsequence $f_{n_k}(x_{n_k}) = f_{n_k}(y_k)$ converges to $f(y)$.

We next show that f is continuous. It suffices to prove that $f(y_k) \to y$ for any point y of E and any sequence (y_k) of points of E with $y_n \to y$. Given such a y and (y_k) we may, by pointwise convergence of (f_n), choose integers n_k such that $|f_{n_k}(y_k) - f(y_k)| < 1/k$; moreover, we may do this in such a way that $n_1 < n_2 < \cdots$. But then $f_{n_k}(y_k) \to f(y)$ by the above remark, and so also $f(y_k) \to f(y)$.

Finally we turn to uniform convergence. If (f_n) does not converge to f uniformly then for some $\epsilon > 0$ we can find an increasing sequence (n_k) of natural numbers and a sequence (y_k) of points of E such that $|f_{n_k}(y_k) - f(y_k)| \geq \epsilon$ for all k. Because E is compact, (y_k) has a convergent subsequence (y_{k_j}), say $y_{k_j} \to x$. Then $f_{n_{k_j}}(y_{k_j}) \to f(x)$ by the observation above and $f(y_{k_j}) \to f(x)$ by continuity of f; this is inconsistent with the fact that $|f_{n_{k_j}}(y_{k_j}) - f(y_{k_j})| \geq \epsilon$ for all j. ■

Remark: The proposition does not hold if the hypothesis of compactness is omitted. For example, take $E = [0, \infty) \subset \mathbb{R}$, $f_n(x) = \min\{x, n\}$, and $f(x) = x$. If $x \in E$ and $x_n \to x$ then $f_n(x_n) \to f(x)$ since if $n > x + 1$ and n is sufficiently large that $x_n < x + 1$, $f_n(x_n) = x_n$. But it is clear that convergence is not uniform, since $|f_n(n+1) - f(n+1)| = 1$ for all n.