SOME SOLUTIONS FOR ASSIGNMENT 10

Before beginning 10.9, it may help to recall exactly what multiple integrals we have defined. For f a continuous function on a k-cell I we defined $\int_I f(\mathbf{x})\,d\mathbf{x}$ by iterated integration (and showed the result was independent of the order of integration); for f a continuous function defined in $\mathbb{R}^k$ with compact support we defined $\int_{\mathbb{R}^n} f(\mathbf{x})\,dx$ as $\int_I f(\mathbf{x})\,d\mathbf{x}$, where I is a k-cell containing $\operatorname{supp} f$. Any other sort of integral must be defined before we can prove anything about it; see the remark at the end of the problem.

10.9: The assignment sheet introduced the notation $I = [0,a]\times[0,2\pi]$, I_0 the interior of I, and then outlined a three-step approach to the problem:

(i) Verify the mapping properties and the formula for the Jacobian as Rudin asks; check also that I_0 maps onto D_0. Then check that, as Rudin indicates, the formula holds if $\operatorname{supp} f \subset D_0$.

As observed by Amanda Hood, this part of the problem is difficult (to say the least) unless one corrects an error in Rudin's statement of the problem: D_0 should be the interior of the D, the disc with radius a and center 0, minus $\{(x,0) \mid 0 \le x < a\}$. Then it is pretty clear that T maps D onto I and is a 1-1 map of I_0 to D_0. The Jacobian of T is

$$J_T(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial y/\partial r \\ \partial x/\partial\theta & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r.$$

Now Theorem 10.9 applies (with $E = I_0$), so that if $f : \mathbb{R} \to \mathbb{R}$ is continuous then

$$\int_{\mathbb{R}^2} f(x,y)\,dx\,dy = \int_{\mathbb{R}^2} f(T(r,\theta))r\,d\theta\,dr, \qquad \text{if } \operatorname{supp} f \subset D_0. \tag{10.1}$$

(ii) Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on I and satisfying $0 \le \phi_n \le 1$, $\operatorname{supp}\phi_n \subset I_0$*, and* $\lim_{n\to\infty}\phi_n = 1$ *on I_0, with uniform convergence on compact subsets of I_0. For $(x,y) \in D$ with $(x,y) = T(r,\theta)$ let $\psi_n(x,y) = \phi_n(r,\theta)$. (T is not 1-1 on D; why is ψ_n well defined?) Then show that*

$$\lim_{n\to\infty}\int_0^a\int_0^{2\pi}\phi_n(r,\theta)f(T(r,\theta))r\,d\theta\,dr = \int_0^a\int_0^{2\pi} f(T(r,\theta))r\,d\theta\,dr, \tag{10.2}$$

and

$$\lim_{n\to\infty}\int_D \psi_n(x,y)f(x,y)\,dx\,dy = \int_D f(x,y)\,dx\,dy. \tag{10.3}$$

Note first that T is not 1-1 on I: $T(0,\theta) = (0,0)$ for all θ, and $T(r,0) = T(r,2\pi) = (r,0)$ for all r. This might present a problem in defining ψ_n as above, because if $T(r_1,\theta_1) = T(r_2,\theta_2)$ then the rule $\psi_n(x,y) = \phi_n(r,\theta)$ whenever $(x,y) = T(r,\theta)$ could be ambiguous. But since $\phi_n = 0$ on the boundary of I this is not a problem: the rule always gives $\psi_n(x,y) = 0$ at such points of possible ambiguity.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $\operatorname{supp} f \subset D$, and suppose $|f| \le M$. We verify (10.2) and (10.3) for this f. It is useful to recall the inequality (discussed in class when we first defined integrals over cells) that if $J = [a_1, b_1] \times [a_2, b_2]$ then

$$\left| \int_J g\, dr\, d\theta \right| \le 2\pi(b_1 - a_1)(b_2 - a)2)a\|g\|. \tag{10.4}$$

To verify (10.2), observe that for any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that if $n \ge N$ then $|\phi_n - 1| \le \epsilon$ on the rectangle $[\epsilon, 1-\epsilon] \times [\epsilon, 2\pi - \epsilon]$. Then, writing

$$g(r,\theta) = \big|\phi_n(r,\theta) f(T(r,\theta)) - f(T(r,\theta))\big| r,$$

we have

$$\left| \int_0^a \int_0^{2\pi} \phi_n(r,\theta) f(T(r,\theta)) r\, d\theta\, dr - \int_0^a \int_0^{2\pi} f(T(r,\theta)) r\, d\theta\, dr \right| \le \int_0^a \int_0^{2\pi} g(r,\theta)\, dr\, d\theta.$$

But $\|g\| \le Ma$ and $\|g\| \le \epsilon Ma$ on $[\epsilon, 1-\epsilon] \times [\epsilon, 2\pi - \epsilon]$, so

$$\begin{aligned}
\int_0^a \int_0^{2\pi} g(r,\theta)\, dr\, d\theta &= \left[\int_0^\epsilon + \int_\epsilon^{a-\epsilon} + \int_{a-\epsilon}^a \right] \int_0^{2\pi} g(r,\theta)\, dr\, d\theta \\
&\le 2(Ma)(2\pi\epsilon) + \int_\epsilon^{a-\epsilon} \left[\int_0^\epsilon + \int_\epsilon^{2\pi-\epsilon} + \int_{2\pi-\epsilon}^{2\pi} \right] g(r,\theta)\, dr\, d\theta \\
&\le 4Ma\pi\epsilon + 2Ma(a - 2\epsilon)\epsilon + (a - 2\epsilon)(2\pi - 2\epsilon)Ma\epsilon \\
&\le 2Ma(2\pi + a + \pi a)\epsilon;
\end{aligned}$$

here we have applied (10.4) repeatedly. Thus (10.2) holds.

The verification of (10.3) is similar. One sees easily that ψ_n converges to 1 uniformly on compact subsets of D_0, so given $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that if $n \ge N$ then $|\phi_n - 1| \le \epsilon$ on the set $K_\delta = \{(x,y) \mid \sqrt{x^2+y^2} \le a - \delta,\ |y| > \delta\} \subset D_0$. Setting

$$h(x,y) = |\psi_n(x,y) f(x,y) - f(x,y)\big|,$$

we have

$$\left| \int_{\mathbb{R}^2} \psi_n(x,y) f(x,y)\, dx\, dy - \int_{\mathbb{R}^2} f(x,y)\, dx\, dy \right| \le \int_{\mathbb{R}^2} h(x,y)\, dx\, dy.$$

Now by our definition, if h is continuous and $\operatorname{supp} h$ is contained in the interior of D then

$$\int_{\mathbb{R}^2} h\, dx\, dy = \int_{-a}^a \int_{-a}^a h(x,y)\, dx\, dy, \tag{10.5}$$

and so

$$\int_{\mathbb{R}^2} h(x,y)\, dx\, dy = \left[\int_{-a}^{-a+\delta} + \int_{-a+\delta}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{a-\delta} + \int_{a-\delta}^{a} \right] \int_{-a}^a h(x,y)\, dx\, dy. \tag{10.6}$$

The first, third, and fifth integrals here are easily estimated, using $|h| \le M$, as $2aM\delta$, $4aM\delta$, and $2aM\delta$, respectively. The second and fourth are treated similarly, we will discuss the fourth. Writing $\alpha(u,y) = \sqrt{u^2 - y^2}$ for $|y| \le u$ we have

$$\int_\delta^{a-\delta}\int_{-a}^{a} h(x,y)\,dx\,dy = \int_\delta^{a-\delta}\left[\int_{-\alpha(a,y)}^{-\alpha(a-\delta,y)} + \int_{-\alpha(a-\delta,y)}^{\alpha(a+\delta,y)} + \int_{\alpha(a-\delta,y)}^{-\alpha(a,y)}\right] h(x,y)\,dx\,dy.$$

The middle integral, where we have $|h| < \epsilon$, is at most $2a^2\epsilon$. Now note that $\alpha(u,y)$ is continuous and hence uniformly continuous on $\{|y| \le u \le a\}$, so that the first and third integrals are each bounded by

$$M\int_\delta^{a-\delta}[u(a,y) - u(a-\delta,y)]\,dy \le Ma\epsilon,$$

if δ is chosen sufficiently small. Thus for such δ the fourth integral in (10.6) is bounded by $(2Ma + 2a^2)\epsilon$; the same bound holds for the second integral. We may suppose also that $\delta \le \epsilon$, in which case we have bounded (10.6) by $(12Ma + 4a^2)\epsilon$, completing the proof of (10.3).

(iii) Construct a sequence ϕ_n as in (iii) and thus prove the theorem.

For $n \ge 2$ let $\chi_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$\chi_n(t) = \begin{cases} 0, & \text{if } t \le 1/2n \text{ or } t \ge 1 - 1/2n, \\ 1, & \text{if } 1/n \le t \le 1 - 1/n \\ 2 - 2nt, & \text{if } 1/2n < t < 1/n, \\ 2 - 2n(1-t), & \text{if } 1 - 1/n < t < 1 - 1/2n; \end{cases}$$

then $\phi_n(r,\theta) = \chi_n(r/a)\chi_n(\theta/(2\pi))$ satisfies the hypotheses in (ii), because if $K \subset I_0$ is compact then $K \subset (\epsilon, 1-\epsilon) \times (\epsilon, 2\pi - \epsilon)$ for some $\epsilon > 0$, so $\phi_n = 1$ on K for sufficiently large n. Now by (10.1), if $f : \mathbb{R} \to \mathbb{R}$ is continuous with $\operatorname{supp} f \subset D$ then

$$\int_0^a\int_0^{2\pi} \phi_n(r,\theta) f(T(r,\theta)) r\,d\theta\,dr = \int_D \psi_n(x,y) f(x,y)\,dx\,dy,$$

so that by (10.2) and (10.3),

$$\int_0^a\int_0^{2\pi} f(T(r,\theta)) r\,d\theta\,dr = \int_D f(x,y)\,dx\,dy. \quad \blacksquare \tag{10.7}$$

Remark: In the above we supposed that $\operatorname{supp} f$ was contained in the interior of D, because only in this case have we defined $\int_D f\,dx\,dy$ (we really defined $\int_{\mathbb{R}^2} f\,dx\,dy$, but it is natural to identify these). If we are given only $f \in \mathcal{C}(D)$, as Rudin suggests, then technically $\int_D f\,dx\,dy$ is as yet undefined (note that defining $f = 0$ outside D may not lead to a function continuous in $\mathbb{R}^2$). However, it is natural in this case to define

$$\int_D f\,dx\,dy = \int_{-a}^{a}\left(\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} f(x,y)\,dx\right)dy,$$

and our proof then applies to show that (10.7) holds for $f \in \mathcal{C}(D)$.