SOME SOLUTIONS FOR ASSIGNMENT 2

7.14: Since $|f| \leq 1$, $|2^{-n}f(3^{2n-1}t)| \leq 2^{-n}$ and the series $\sum_{n=1}^{\infty} 2^{-n}f(3^{2n-1}t)$ defining $x(t)$ converges uniformly on $\mathbb{R}$ by the Weierstrass M test, with sum between 0 and 1; since each term of this series is continuous, so is $x(t)$, and since $f \geq 0$ and $\sum_{n=1}^{\infty} 2^{-n} = 1$, $0 \leq x(t) \leq 1$. The same argument shows that $y(t)$ is continuous and $0 \leq y(t) \leq 1$. Since each component of Φ is continuous, so is Φ, and $\Phi(t) \in [0,1]^2$.

To understand the rest of the problem one needs the following

Proposition: *Fix an integer $p \geq 2$. Then every real number $t \geq 0$ has a p-*ary expansion*, or* expansion with base *p: for some integer r,*

$$t = \sum_{i=r}^{\infty} a_i p^{-i}, \qquad a_i \in \mathbb{Z}, \quad 0 \leq a_i \leq p-1. \tag{2.1}$$

Moreover, every sequence $(a_i)_{i=r}^{\infty}$ with $0 \leq a_i \leq p-1$ defines a real number $t \in I$ through (2.1).

We won't prove this. Everyone is familiar with the *decimal* (base 10) expansion of a real number; the expansions with bases 2 and 3 are called the *binary* and *ternary* expansions, respectively. When p is understood, the p-ary expansion (2.1) is frequently written $a_r \cdots a_0.a_1a_2a_3\cdots$ (if $r \leq 0$) or $0.0\cdots 0a_ra_{r+1}\cdots$ (if $r \geq 1$). The expansion of t is not unique, for two reasons. First, given any expansion we may obtain others by decreasing r and introducing "leading zeros;" this causes no confusion. Further, if $t = mp^{-j}$ for some $j, m \in \mathbb{N}$, t has two expansions, one in which $a_i = 0$ for sufficiently large i, one in which $a_i = p-1$ for sufficiently large i. For example, the real number with unique decimal expansion $0.33333\cdots$ has unique binary expansion $0.010101010\cdots$ and ternary expansions $0.10000\cdots$ and $0.022222\cdots$.

Now suppose that $(x_0, y_0) \in I^2$; x_0 and y_0 have binary expansions $x_0 = \sum_{n=1}^{\infty} b_n 2^{-n}$, $y_0 = \sum_{n=1}^{\infty} c_n 2^{-n}$. If we define (a_i) by $a_{2n-1} = b_n$, $a_{2n} = c_n$ then we obtain the representation of (x_0, y_0) given in the text. On the other hand, we will discuss points t in the *domain* I of Φ in terms of their *ternary* representations. In particular, let C denote the set of all $t_0 \in I$ having a ternary representation $t_0 = \sum_{i=1}^{\infty} d_i 3^{-i}$ in which none of the digits d_i are equal to 1; C **is the Cantor set**. We may alsofollow Rudin and write the expansion of $t_0 \in C$ as $t_0 = \sum_{i=0}^{\infty}(2a_i)3^{-i-1}$, where $a_i = d_{i+1}/2$. Rudin assumes that the first ternary digit of t_0 is 0, but that is not needed.

Now we prove that if $t_0 \in C$ with $t_0 = 0.d_1d_2d_3\cdots = 0.(2a_0)(2a_1)(2a_2)\cdots$ then $\Phi(t_0) = (x_0, y_0)$ with $x_0 = 0.a_1a_3a_5\cdots$ and $y_0 = 0.a_2a_4a_6\cdots$; in view of the above comments, this implies that $\Phi(C) = I^2$, so certainly $\Phi(I) = I^2$. If we show that $f(3^k t_0) = a_k$ for $k \geq 1$ then we are done (just plug this into the formulas for $x(t)$ and $y(t)$). Now $3^k t_0 = d_1d_2\cdots d_k.d_{k+1}d_{k+2}\cdots$; the integer part $d_1d_2\cdots d_k$ of this number is even, since each d_i is even, and by the property $f(t+2) = f(t)$ we have that $f(3^k t_0) = f(0.d_{k+1}d_{k+2}\cdots)$. But if $d_{k+1} = 0$ then

$$0 \leq 0.d_{k+1}d_{k+2}\cdots \leq 0.02222\cdots = 1/3,$$

so that $f(3^k t_0) = f(0.d_{k+1}d_{k+2}\cdots) = 0 = a_k$, from the choice of f. Similarly, if $d_{k+1} = 2$ then

$$2/3 = 0.20000\cdots \leq 0.d_{k+1}d_{k+2}\cdots \leq 0.22222\cdots = 1,$$

so that $f(3^k t_0) = f(0.d_{k+1}d_{k+2}\cdots) = 1 = a_k$. ■