SOME SOLUTIONS FOR ASSIGNMENT 3

3.A: A function f defined on $\mathbb{R}$ is called *periodic with period* T if $f(x+T) = f(x)$ for all $x \in \mathbb{R}$. Let $\mathcal{C}_T(\mathbb{R}) = \{ f \in \mathcal{C}(\mathbb{R}) \mid f \text{ is periodic with period } T \}$. In general, $\mathcal{C}_T(\mathbb{R})$ might be defined to consist of real valued or of complex valued functions; in this problem we consider only real valued functions.

(a) Let $\mathcal{A} \subset \mathcal{C}_T(\mathbb{R})$ be an algebra (over $\mathbb{R}$) such that $\mathcal{A}$ vanishes at no point and such that if x and y are two points of $\mathbb{R}$ with $x - y$ not an integer multiple of T, then there is an $f \in \mathcal{A}$ with $f(x) \neq f(y)$. Prove: the uniform closure of $\mathcal{A}$ is $\mathcal{C}_T(\mathbb{R})$.

(b) Prove that the set of all finite linear combinations of the functions 1, $\sin nx$, $n \in \mathbb{N}$, and $\cos nx$, $n \in \mathbb{N}$, is dense in $\mathcal{C}_{2\pi}(\mathbb{R})$.

Solution: (a) We follow the given hint and let $\phi : \mathbb{R} \to S^1$ be defined by $\phi(x) = e^{2\pi i x/T}$. We accept the fact that this map is onto and that $\phi(x) = \phi(y)$ iff $y - x = kT$ for some $k \in \mathbb{Z}$ (this was proved rigorously in class somewhat after this assignment). For $f \in \mathcal{C}(S^1)$ define a function $\Phi(f)$ on $\mathbb{R}$ by $(\Phi(f))(x) = f(\phi(x))$; then $\Phi(f)$ is continuous (as the composition of continuous functions) and periodic with period T (because ϕ is); thus $\Phi : \mathcal{C}(S^1) \to \mathcal{C}_T(\mathbb{R})$. Moreover, one easily sees that Φ is an algebra homomorphism:

$$\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g) \qquad \text{and} \qquad \Phi(fg) = \Phi(f)\Phi(g).$$

Now observe that, if $\|\cdot\|$ denotes the uniform norm, then for $x \in \mathbb{R}$, $|\Phi(f)(x)| = |f(\phi(x))| \leq \|f\|$, so $\|\Phi(f) \leq \|f\|$. Similarly, $\|f\| \leq \|\Phi(f)\|$, because $|f(\phi(x))| \leq \|\Phi(f)\|$ and since ϕ is onto we have $\|f\| = \sup_x |f(\phi(x))|$. Thus Φ is an isometry. In particular, this means that Φ is one-to-one, since if $\Phi(f) = \Phi(g)$ then $\|f - g\| \leq \|\Phi(f) - \Phi(g)\| = 0$, so $f = g$.

We next verify that Φ is onto. The restriction of ϕ to the interval $[0, T)$ is clearly a bijection of that interval with S^1; we let $\psi : S^1 \to [0, T)$ denote the inverse of this map and for $f \in \mathcal{C}_T(\mathbb{R})$ define $\Psi(f) : S^1 \to \mathbb{R}$ by $(\Psi(f))(z) = f(\psi(z))$. We will see shortly that $\Psi(f)$ is continuous; thus $\Psi : \mathcal{C}_T(\mathbb{R}) \to \mathcal{C}(S^1)$. Now for $f \in \mathcal{C}_T(\mathbb{R})$, $f = \Phi(\Psi(f))$ since if $x \in [0, T)$, $\Phi(\Psi(f))(x) = f(\psi(\phi(x)) = f(x)$, and since we know that $\Phi(\Psi(f))$ is T-periodic and agrees with f on $[0, T)$, it must agree with f everywhere. This proves that Φ is onto.

We must still show that if $f \in \mathcal{C}_T(\mathbb{R})$ then $\Psi(f)$ is continuous. Now $\Psi(f)$ is clearly continuous at all points where ψ is continuous, that is, all points except $z = 1$. One way to check continuity there is to observe that we could have made the same construction as above starting with the inverse of the restriction of ϕ to $[-T/2, T/2)$ rather than to $[0, T)$; the resulting map, say $\tilde{\Psi}$, from $\mathcal{C}_T(\mathbb{R})$ to $\mathcal{C}(S^1)$ would in fact coincide with Ψ and $\tilde{\Psi}(f)$ would clearly be continuous at $z = 1$.

With all this machinery, the proof is easy. $\Phi^{-1}(\mathcal{A}) \subset \mathcal{C}(S^1)$ is easily seen to be an algebra which separates points and vanished at no point, so $\overline{\Phi^{-1}(\mathcal{A})} = \mathcal{C}(S^1)$ by the Stone-Weierstrass Theorem. Because Φ is a bijective isometry, $\mathcal{C}(S^1) = \overline{\Phi^{-1}(\mathcal{A})} = \Phi^{-1}(\overline{\mathcal{A}})$ and applying Φ to both sides of this equation we have $\mathcal{C}_T(\mathbb{R}) = \overline{\mathcal{A}}$.

(b) The set of all finite linear combinations

$$a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

is an algebra; this follows from such trigonometric identities as

$$\cos(A)\cos(B) = (1/2)\left[\cos(A+B) + \cos(A-B)\right].$$

It vanishes at no point because it contains all constant functions, and separates points because if $0 \leq a < b < 2\pi$ then either $\sin a \neq \sin b$ or $\cos a \neq \cos b$.