SOME SOLUTIONS FOR ASSIGNMENT 5

8.6. (Slightly rephrased.) Suppose that $f:\mathbb{R}\to\mathbb{C}$, that

$$f(x)f(y)=f(x+y), \tag{5.1}$$

for all real x and y, and that f is not identically zero. Prove that there is a $c\in\mathbb{C}$ such that $f(x)=e^{cx}$ for all $x\in\mathbb{R}$, assuming: (a) that f is differentiable; (b) only that f is continuous.

Solution: We are told that f is not identically zero; suppose that $f(x_0)\neq 0$. Taking $y=0$ and $x=x_0$ in (5.1), we find that $f(0)=1$, and taking then $y=-x$ we find that $f(-x)=1/f(x)$; in particular, $f(x)$ is nonzero for all x. Moreover, by induction on n we find from (5.1) that

$$f(nx)=f(x)^n, \qquad x\in\mathbb{R},\ n\in\mathbb{N}. \tag{5.2}$$

(a) Since f is differentiable we may differentiate (5.1) with respect to y and then set $y=0$ to obtain $f'(x)=cf(x)$, where $c=f'(0)$. But then if $g(x)=e^{-cx}f(x)$ we find that $g'(x)=0$ for all x; thus g is a constant (by the mean value theorem or 5.11(b)) and since $g(0)=1$, $g(x)=1$ for all x, i.e., $f(x)=e^{cx}$.

(b) As a warm-up, let us do the case in which f is real-valued. From $f(x)=f(x/2)^2$ we then see that $f(x)>0$ for all x, and from Theorem 8.6 it follows that $f(1)=e^c$ for some unique real number c. We now show that

$$f(x)=e^{cx}. \tag{5.3}$$

for all $x\in\mathbb{R}$; we already know that this holds for $x=0$ and $x=1$. For $n\in\mathbb{N}$ the two numbers $f(1/n)$ and $e^{c/n}$ are both positive n^{th} roots of e^c, and so must be equal (Theorem 1.21), establishing (5.3) for $x=1/n$; but then $f(m/n)=f(1/n)^m=(e^{c/n})^m=e^{cm/n}$, so that (5.3) holds for positive rationals. From $f(-x)=1/f(x)$ it follows that (5.3) holds for all rationals. But both sides of (5.3) are continuous functions of x; if they agree on the dense set $\mathbb{Q}$ they must agree for all x.

Now suppose that f may take complex values. The difficulty then is that the equation $f(x)=f(x/n)^n$ does not determine $f(x/n)$ in terms of $f(x)$, since a complex number has n (complex) n^{th} roots. When f was real and hence positive we used the fact that every positive number has a unique *positive* n^{th} root; we need a substitute for this, and we find it in the continuity of f. We will use freely the polar representation $z=re^{i\theta}=r(\cos\theta+i\sin\theta)$ of a complex number z; properties of this can be derived from our work on expoentials, logarithms, and trig functions, but we omit details.

The right half plane $H=\{z\mid \operatorname{Re}z>0\}$ is an open set containing $f(0)=1$, so by the continuity of f there exists an $r>0$ such that if $|x|<r$ then $f(x)\in D$. Choose n sufficiently large that $2^{-n}<r$, so that $f(2^{-n})\in H$; $f(2^{-n})$ thus has a representation $f(2^{-n})=\rho e^{i\phi}=e^{\log\rho+i\phi}$ with $\rho>0$ and $|\theta|<\pi/2$. Now consider $f(2^{-(n+1)})$ by ; this number must lie in H (since $2^{-(n+1)}<2^{-n}<r$) and satisfy $f(2^{-(n+1)})^2=f(2^{-n})$ by (5.2); these conditions uniquely determine that $f(2^{-(n+1)})=\sqrt{\rho}e^{i\phi/2}=e^{(\log\rho+i\phi)/2}$. [**Note**: this is the key idea, continuity tells us that $f(2^{-(n+1)})\in H$ and this determines which of the two possible square roots of $f(2^{-n})$ it is.] Repeating this argument we find inductively that for $j\geq 0$, $f(2^{-(n+j)})=e^{2^{-j}(\log\rho+i\phi)}$, and then from (5.2) that for x any dyadic rational ($x=k2^{-m}$, $m\in\mathbb{N}$, $k\in\mathbb{Z}$) that $f(x)=e^{x2^n(\log\rho+i\phi)}=e^{cx}$ with $c=2^n(\log\rho+i\phi)$. Since the dyadic rationals are dense in $\mathbb{R}$ we get $f(x)=e^{cx}$ for all x in $\mathbb{R}$, by the continuity argument used above.